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PREFACE

This volume contains a collection of papers presented at the Nankai Symposium on "Lattice Statistics and Mathematical Physics," which was organized to honor the seventieth birthday of Professor Fa Yueh Wu (伍法岳). This conference took place at the Nankai Institute of Mathematics in Tianjin, China, hosted by its Vice Director Professor Mo-Lin Ge, October 7–11, 2001, co-organized with APCTP and Beijing Normal University.

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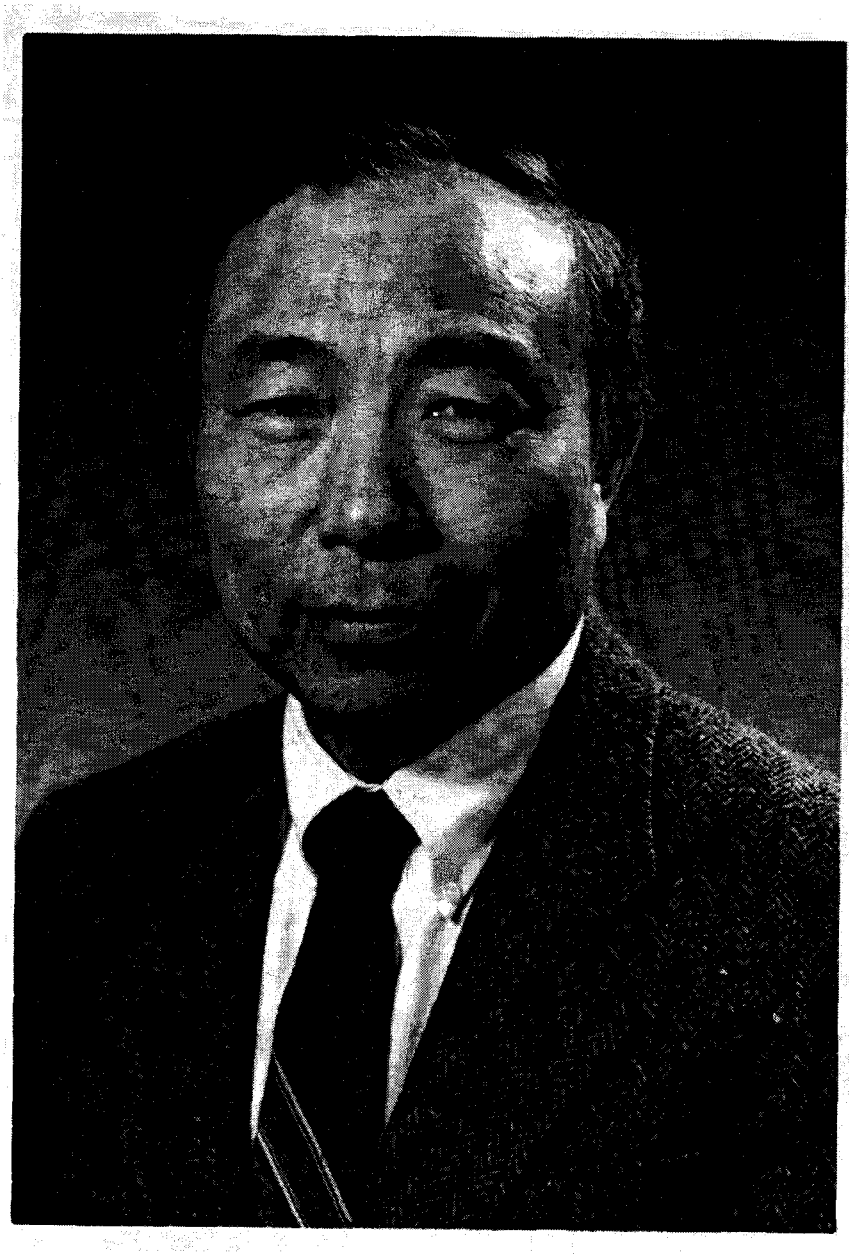
Jacques H. H. Perk
Mo-Lin Ge
May 2002



Fa-Yueh Wu (伍法岳)

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FA YUEH WU* (伍法岳)

1. Brief Biography

Fa-Yueh Wu (a.k.a. Fred Wu) was born January 1932. He moved with his parents and the Chinese government throughout the Sino-Japanese war and the civil war from 1938 to 1949. It may be noted that he graduated from Nankai Junior High School in Chungking, making him an "alumnus of Nankai." After graduating from high school in 1949, he eventually moved with his parents to Taiwan.

There he entered the Chinese Naval College of Technology in 1949, obtaining a B.S. degree in Electrical Engineering in 1954, and receiving the commission as an Ensign in the navy.

Wu was sent by the Chinese navy to the U.S. in 1955 to receive training at the Naval School of Electronics in San Francisco and the Instructors' School in San Diego, returning to Taiwan in 1956 to teach Electronics at the Naval Academy. He was a full-fledged expert on radar and sonar at that time, with a skill he has found useful recently in resoldering and fixing his broken remote car key.

Fred Wu was (and probably still is) a good player of Chinese chess. He was a regional champion in Taiwan in 1951, and later the 1956 champion of all armed forces in Taiwan while a naval ensign. His favorite pastime in his graduate student years was to play chess "blind" with classmates while working on his homework. He has challenged the participants of the symposium to see if he is still as sharp as he used to be. But nobody took up the challenge.

The very next year, in 1957, he entered the graduate school of the National Tsing Hua University in Taiwan, obtaining an M.S. degree in physics two years later.

In 1959 he entered Washington University in St. Louis as a physics graduate student, where he studied under the late Professor Eugene Feenberg, working on many-body problems and obtaining his Ph.D. in physics in 1963. He taught for four years at Virginia Polytechnic Institute before coming to Northeastern University in 1967, where he is presently the Matthews University Distinguished Professor of Physics.

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Wu has accumulated a publication list[a] of over 200 papers and monographs. His earliest paper is in Chinese and published by him in 1955 while an Ensign in the Chinese navy. This paper[1] bears the title "On the discussions of 'free waveforms'." While Wu is known mostly for his publications in statistical mechanics, his works on many-body problems, especially those on liquid helium, have also been influential for many years, see e.g. Ref. 3, which was part of his Ph.D. thesis research. He has even published one experimental paper[5] with the title "Four slow neutron converters."

Wu came to Northeastern to work with Elliott Lieb in 1967, and in 1968 they published a classic joint paper on the ground state of the Hubbard model.[11] This paper has become prominent in the theory of high-T_c superconductors. Anderson has attributed to it "predicting" the existence of quarks, in his Physics Today article on the Centennial of the discovery of electrons. Lieb and Wu also wrote a monograph on vertex models in 1970, which has become a principal reference in the field for decades.[30]

Since Wu came to the U.S. in 1959 as an ensign in the Chinese navy and was not decommissioned then, he was promoted in rank while a graduate student and a faculty member, eventually reaching the rank of Lieutenant in 1963. Therefore, much of his early work including the monograph with Lieb was done by a Lieutenant of the Chinese navy. Eventually, he could not be promoted further since for that he had to take an exam and the Navy was not sure whether he could pass it. He was later decommissioned from the rank of Navy Lieutenant in 1971. Thus the Chinese Navy saved a bunch of retirement benefits paid to retirees depending on the length of their service.

Wu has worked on a wide-range of topics in many-body theory and statistical mechanics, including contributions in lattice statistics, graph theory, combinatorics, number theory, knot theory, and the interrelation between these topics.

Wu's 1982 review on the Potts model is also well-known.[88] This paper has been receiving over a hundred citations for many years ever since it was published.[b] In 1992 Wu published another well-received review on knot theory.[153] Fred Wu has since been referred to as being "knotty" by Professor Lebowitz, which might be said to be a little "naughty" of Joel.

2. Some other selected publications

Another classic is the paper on the Free Fermion Model.[16] This was later extended to its checkerboard version during one of Wu's many visits to Taiwan.[49,51] Fred Wu was a close friend of the late Professor Piet Kasteleyn, who was co-advising my thesis work with Professor Hans Capel in Leiden at the time. Kasteleyn noted the

[a]This list has been appended and a selection of this work has been cited in the following, reflecting the taste of the present editor.

[b]In 1982, the year the Potts review was published, it was the fifth most-cited paper among papers published in all of physics according to E. Garfield, [Current Comments **48**, 3 (1984)].

similarity of my first major paper[c] on the alternating XY-chain and the preprints of the above works. Both showed multiple phase transitions.

Well-known is also the Baxter–Wu Model, i.e. the Ising model with three-spin interactions on a triangular lattice.[44,48] Another classic paper, by Baxter, Kelland and Wu, concerns the graphical construction of the equivalence of the partition functions of the Potts model and a certain staggered six-vertex model.[56] Many people consider this construction easier than the algebraic method of Temperley and Lieb. Both methods are widely used these days. This paper is also at the basis of my first joint work with Fred Wu.[102] Here we generalized this equivalence to include the nonintersecting string (NIS) model of Stroganov and Schultz, alias the Close-Packed Loop Model.

The six-vertex model is boundary-condition dependent. However, Brascamp, Kunz and Wu established for the first time that at sufficiently low temperatures or sufficiently high fields the six-vertex models with either periodic or free boundary conditions are equivalent.[42]

Another remarkable result of Wu is that a very general staggered eight-vertex model in the Ising language (introduced in 1971 by Kadanoff and Wegner and by Wu[28] in two back-to-back papers), but with the special magnetic field $i\pi k_{\rm B}T/2$ of Lee and Yang added, is equivalent to Baxter's symmetric eight-vertex model and hence solvable.[104] The general eight-vertex model without this field is not known to be solvable.

The dimer model on the honeycomb lattice was first solved by Kasteleyn. This has recently been generalized by Huang, Wu, Kunz and Kim to the case where the dimers have nearest-neighbor interaction.[172] This model relates to a degenerate case of the six-vertex model, requiring a special Bethe Ansatz analysis. The resulting phase diagram of this five-vertex model is quite complicated. This work has also been used in papers by Huang, Popkov and Wu on the three-dimensional dimer model.[175,180] Its phase diagram is also quite complex.

In 1999, Lu and Wu initiated work on dimer and Ising models on nonorientable surfaces,[191,200,202] and generalized a reciprocity theorem in dimer combinatorics due to R. Stanley and J. Propp.[203] There is now much activity in this area, inspired by this work, as there is much interest in finite-size corrections and conformal field theories on more complicated surfaces.

This is, of course, only a limited selection. A more precise understanding of the impact of Wu's work can be obtained by going over the following publication list and from the many papers in the volume. Therefore, I can speak on behalf of the other editor Professor Ge and the many participants of the symposium: Happy birthday and thank you, Professor Wu, for your many special insights and for being a friend of us all and not just a colleague.

[c] J.H.H. Perk, H.W. Capel, M.J. Zuilhof, and Th.J. Siskens, Physica A **81**, 319–348 (1975).

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CONTENTS

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STAR-TRIANGLE EQUATIONS AND IDENTITIES IN HYPERGEOMETRIC SERIES

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In this paper, we introduce the cyclic basic hypergeometric series ${}_{p+1}\Phi_p$ with $q \to \omega$ where $\omega^N = 1$. This is a terminating series with N terms, whose summand has period N. We show how the Fourier transform of the weights of the integrable chiral Potts model are related to the ${}_2\Phi_1$, which is summable. We show that ${}_3\Phi_2$ satisfies certain transformation formulae. We then show that the Saalschützian ${}_4\Phi_3$ series is summable at argument $z = \omega$. This then gives the simplest proof of the star-triangle relation in the chiral Potts model. Finally, we let $N \to \infty$, where the star-triangle equation becomes a two-sided identity for the hypergeometric series.

1. Introduction

1.1. *Definitions*

The generalized hypergeometric series is defined[1,2] as

$$ {}_{p+1}\mathrm{F}_p\left[\begin{matrix} a_1,\cdots,a_{p+1} \\ b_1,\cdots,b_p \end{matrix}; z\right] = \sum_{l=0}^{\infty} \frac{(a_1)_l \cdots (a_{p+1})_l}{(b_1)_l \cdots (b_p)_l \, l!} z^l, \tag{1} $$

where

$$ (a)_l = \Gamma(a+l)/\Gamma(a) = a(a+1)\cdots(a+l-1), \tag{2} $$

while the basic hypergeometric hypergeometric series is

$$ {}_{p+1}\Phi_p\left[\begin{matrix} \alpha_1,\cdots,\alpha_{p+1} \\ \beta_1,\cdots,\beta_p \end{matrix}; z\right] = \sum_{l=0}^{\infty} \frac{(\alpha_1;q)_l \cdots (\alpha_{p+1};q)_l}{(\beta_1;q)_l \cdots (\beta_p;q)_l (q;q)_l} z^l, \tag{3} $$

in which

$$ (x;q)_l \equiv \begin{cases} (1-x)(1-xq)\cdots(1-xq^{l-1}), & l \geq 0, \\ 1/[(1-xq^{-1})(1-xq^{-2})\cdots(1-xq^{l})], & l < 0. \end{cases} \tag{4} $$

The hypergeometric series ${}_{p+1}\mathrm{F}_p$ can be obtained from ${}_{p+1}\Phi_p$ by taking the limit

$$ q \to 1 \quad \text{with} \quad \alpha = q^a, \quad (\alpha;q)_l/(q;q)_l \to (a)_l \,. \tag{5} $$

1.2. *Known Identities*

A hypergeometric series is summable if the series can be written in terms of ratios of products of Gamma functions, while for the summable basic series it is written in terms of the q-products defined in (4). The most well-known summation formula is due to Gauss

$$ {}_2\mathrm{F}_1\left[\begin{matrix}a,b\\c\end{matrix};1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{6} $$

which is a summation formula for ${}_2\mathrm{F}_1$ of unit argument. The other is Saalschütz's theorem

$$ {}_3\mathrm{F}_2\left[\begin{matrix}a,b,-n\\c,d\end{matrix};1\right]=\frac{(c-a)_n(c-b)_n}{(c-a-b)_n(c)_n},\quad \text{for}\quad c+d=a+b-n+1\,, \tag{7} $$

for a terminating Saalschützian ${}_3\mathrm{F}_2$ of unit argument. In general, a series is called Saalschützian if it satisfies the Saalschütz condition

$$ 1+a_1+\cdots+a_{p+1}=b_1+\cdots+b_p\,. \tag{8} $$

Most of the summation formulae for the usual ${}_{p+1}\mathrm{F}_p$ hypergeometric series have basic series analogues.[1,2] The summability condition on the argument of $z=1$ for the hypergeometric series must be replaced by $z=q$ for the basic series, while the Saalschütz condition is seen from (5) to become

$$ q\alpha_1\cdots\alpha_{p+1}=\beta_1\cdots\beta_p, \tag{9} $$

as a and b are the exponents of α and β. As an example, Dougall's theorem summing a terminating ${}_7\mathrm{F}_6$ of unit argument generalizes to Jackson's theorem for terminating ${}_8\Phi_7$ of argument $z=q$.

We shall now show that the basic hypergeometric series at root of unity are intimately related to the integrable chiral Potts model. Indeed, many of the results presented here are implicit in the earlier works.[3–13] Since the notations used in several of these works[6–11] are unconventional, making the connections obscure, we present here the results in more standard notation.

2. The Cyclic Basic Hypergeometric Series

2.1. *Definitions*

Since most of the summation formulae are valid only for terminating series, it is straightforward to analytically continue q to a root of unity without any convergence problems. For $q\to\omega\equiv e^{2\pi i/N}$, we find

$$ (x;\omega)_{l+N}=(1-x^N)(x;\omega)_l,\qquad (\omega;\omega)_{l+N}=0, \tag{10} $$

$$ (x;\omega)_{-l}=\omega^{\frac{1}{2}l(l+1)}/[(-x)^l(\omega x^{-1};\omega)_l], \tag{11} $$

$$ (x;\omega)_{l+k}=(x;\omega)_k\,(\omega^k x;\omega)_l. \tag{12} $$

From (10), we see immediately that the basic series is ill-defined for $q = \omega$ unless $\alpha_{p+1} = \omega^{-J}$ for some $J < N$. We shall here restrict ourselves to the case with $\alpha_{p+1} = \omega^{-N+1} = \omega$, so that there are N terms in the series.

Definition: A cyclic basic hypergeometric series is a terminating series of N terms

$$ {}_{p+1}\Phi_p \left[\begin{matrix} \omega, \alpha_1, \cdots, \alpha_p \\ \beta_1, \cdots, \beta_p \end{matrix} ; z \right] = \sum_{l=0}^{N-1} \frac{(\alpha_1;\omega)_l \cdots (\alpha_p;\omega)_l}{(\beta_1;\omega)_l \cdots (\beta_p;\omega)_l} z^l, \tag{13} $$

whose summand is periodic in N.

Using (10), we find that the requirement for periodic summand is satisfied if

$$ z^N = \prod_{j=1}^{p} \frac{1-\beta_j^N}{1-\alpha_j^N}. \tag{14} $$

Unlike the ordinary basic hypergeometric series in (3), where the dependences on the parameters α_l and β_l are elementary, the periodicity requirement makes the dependences on these parameters very complicated, with an extremely complex N-sheeted Riemann sheet structure.

Because of this periodicity, we may change the indices of the summation $l \to -l$ in (13) and then let $l \to l + k$ while using (11), to find

$$ {}_{p+1}\Phi_p \left[\begin{matrix} \omega, \alpha_1, \cdots, \alpha_p \\ \beta_1, \cdots, \beta_p \end{matrix} ; z \right] = {}_{p+1}\Phi_p \left[\begin{matrix} \omega, \omega\beta_1^{-1}, \cdots, \omega\beta_p^{-1} \\ \omega\alpha_1^{-1}, \cdots, \omega\alpha_p^{-1} \end{matrix} ; \frac{\beta_1 \cdots \beta_p}{z\alpha_1 \cdots \alpha_p} \right] \tag{15} $$

$$ = {}_{p+1}\Phi_p \left[\begin{matrix} \omega, \omega^k\alpha_1, \cdots, \omega^k\alpha_p \\ \omega^k\beta_1, \cdots, \omega^k\beta_p \end{matrix} ; z \right] \frac{(\alpha_1;\omega)_k \cdots (\alpha_p;\omega)_k}{(\beta_1;\omega)_k \cdots (\beta_p;\omega)_k} z^k. \tag{16} $$

Since $\alpha_{p+1} = \omega$, the series in (13) is called a Saalschützian if

$$ \omega^2 \alpha_1 \alpha_2 \cdots \alpha_p = \beta_1 \beta_2 \cdots \beta_p, \qquad z = \omega. \tag{17} $$

Clearly, if the left-hand side of (15) is a Saalschützian, so is the right-hand side, and vice versa.

2.2. *Cyclic basic series* ${}_2\Phi_1$

It has been found[3,15] that the Boltzmann weights of the integrable chiral Potts model can be written in product form, i.e.

$$ W(n) = \gamma^n \frac{(\alpha;\omega)_n}{(\beta;\omega)_n} = \left(\frac{\beta}{\gamma\alpha}\right)^{-n} \frac{(\omega/\beta;\omega)_{-n}}{(\omega/\alpha;\omega)_{-n}} = W^*(-n), \quad \gamma^N = \frac{1-\beta^N}{1-\alpha^N}. \tag{18} $$

The Boltzmann weight of an edge connecting spin a and spin b is chiral, namely, $W(a-b) \neq W(b-a)$, and arrows are introduced to indicate the direction from spin a to b, as shown in Fig. 1. Here we have introduced $W^*(a-b) = W(b-a)$ to indicate the operation of arrow-reversing.

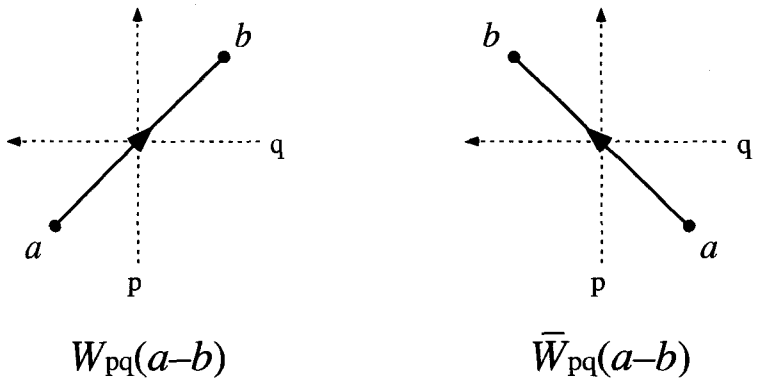

Fig. 1. Boltzmann Weights

Since the weights are periodic with period N, their Fourier transforms may be written as

$$W^{(f)}(k) = \sum_{n=0}^{N-1} \omega^{nk} W(n) = {}_2\Phi_1 \left[\begin{matrix} \omega, \alpha \\ \beta \end{matrix} ; \gamma \omega^k \right]. \tag{19}$$

2.2.1. *Recursion Formula*

It has been found originally in Canberra[3] that

$$\frac{W^{(f)}(k)}{W^{(f)}(0)} = {}_2\Phi_1 \left[\begin{matrix} \omega, \alpha \\ \beta \end{matrix} ; \gamma\omega^k \right] {}_2\Phi_1 \left[\begin{matrix} \omega, \alpha \\ \beta \end{matrix} ; \gamma \right]^{-1} = \frac{(\omega/\beta)^k (\gamma;\omega)_k}{(\omega\alpha\gamma/\beta;\omega)_k} = \frac{(\beta/\gamma\alpha;\omega)_{-k}}{\alpha^k (\omega/\gamma;\omega)_{-k}}. \tag{20}$$

The proof of this recursion relation has been given in our Taniguchi lectures.[4] This was later extended to a more general case by Kashaev et al.[7]

$${}_2\Phi_1 \left[\begin{matrix} \omega, \omega^m\alpha \\ \omega^n\beta \end{matrix} ; \gamma\omega^k \right] \Big/ {}_2\Phi_1 \left[\begin{matrix} \omega, \alpha \\ \beta \end{matrix} ; \gamma \right] = \frac{(\omega/\beta)^k}{(\gamma\omega^k)^n} \frac{(\beta;\omega)_n (\gamma;\omega)_k (\omega\alpha/\beta;\omega)_{m-n}}{(\alpha;\omega)_m (\omega\alpha\gamma/\beta;\omega)_{m-n+k}}. \tag{21}$$

2.2.2. *Baxter Formula*

Consider the determinant whose elements are the weights in (18), i.e.

$$D = \det_{1 \le l,k \le N} W(l-k), \tag{22}$$

Baxter gave the following formula[5] without proof:

$$D = \Phi_0^N N^{\frac{1}{2}N} \prod_{j=1}^{N-1} \left[\frac{(\alpha - \omega^{-1-j}\beta)}{(1-\omega^{-1-j}\beta)(1-\omega^j\alpha)} \right]^j, \tag{23}$$

where

$$\Phi_0 \equiv e^{i\pi(N-1)(N-2)/12N}. \tag{24}$$

A detailed proof was subsequently given by us in Ref. 14. Since this is a cyclic determinant, we find

$$D = \prod_{j=0}^{N-1} W^{(f)}(j) = [W(0)^{(f)}]^N \prod_{j=1}^{N-1} \left[\frac{W^{(f)}(j)}{W^{(f)}(0)}\right]. \tag{25}$$

Baxter's formula (23) and the recursion formula (20) may now be used to prove the following theorem.

Theorem 1: Every cyclic basic hypergeometric series ${}_2\Phi_1$ is summable, and is given by

$${}_2\Phi_1\left[\begin{matrix}\omega,\alpha\\ \beta\end{matrix};\gamma\right] = \omega^{\ell} N^{\frac{1}{2}} \Phi_0 \left(\frac{\omega}{\beta}\right)^{\frac{1}{2}(N-1)} \frac{p(\omega\alpha/\beta)p(\gamma)}{p(\alpha)p(\omega/\beta)p(\omega\alpha\gamma/\beta)}, \tag{26}$$

where ℓ takes N different integer values for the N different Riemann sheets, and

$$p(\alpha) = \prod_{j=1}^{N-1} (1-\omega^j\alpha)^{j/N}. \tag{27}$$

Here summable mean that the series is expressible as products. It is worthwhile to emphasize that the basic hypergeometric function ${}_2\Phi_1$ is an N-valued function of α and β with a complicated Riemann surface. The function $p(\alpha)$ has $N-1$ branch points at $\alpha = \omega^j$ for $j = 1, \cdots, N-1$. Due to the appearance of the composite functions—particularly, $p(\gamma)$ with γ found from (18) to be an N-valued function of α and β—we can see that it is non-trivial to describe the Riemann surface. It is rather amazing even to us that Baxter and others (see Refs. 17,18 and citations quoted there) have somehow found a way out without the detailed knowledge of the Riemann surfaces.

2.3. *Transformation formula for* ${}_3\Phi_2$

We shall now derive a transformation formula for the cyclic basic series ${}_3\Phi_2$. Using the convolution theorem, we may write

$$\begin{aligned} {}_3\Phi_2\left[\begin{matrix}\omega,\alpha_1,\alpha_2\\ \beta_1,\beta_2\end{matrix};\gamma\right] &= \sum_{l=0}^{N-1} \left[\frac{(\alpha_1;\omega)_l}{(\beta_2;q)_l} u^l\right] \left[\frac{(\alpha_2;q)_l}{(\beta_1;q)_l}\left(\frac{z}{u}\right)^l\right] \\ &= N^{-1} \sum_{k=0}^{N-1} {}_2\Phi_1\left[\begin{matrix}\omega,\alpha_1\\ \beta_2\end{matrix};\omega^{-k}u\right] {}_2\Phi_1\left[\begin{matrix}\omega,\alpha_2\\ \beta_1\end{matrix};\frac{\omega^k\gamma}{u}\right], \end{aligned} \tag{28}$$

where

$$u^N = \frac{1-\beta_2^N}{1-\alpha_1^N}, \qquad \frac{z^N}{u^N} = \frac{1-\beta_1^N}{1-\alpha_2^N}. \tag{29}$$

Now we can use the recursion formula (20) to obtain

$$
{}_3\Phi_2\left[\begin{matrix}\omega,\alpha_1,\alpha_2\\ \beta_1,\beta_2\end{matrix};z\right] = N^{-1}\,{}_2\Phi_1\left[\begin{matrix}\omega,\alpha_1\\ \beta_2\end{matrix};u\right]{}_2\Phi_1\left[\begin{matrix}\omega,\alpha_2\\ \beta_1\end{matrix};\frac{z}{u}\right]
$$
$$
\times\,{}_3\Phi_2\left[\begin{matrix}\omega,z/u,\beta_2/\alpha_1 u\\ \omega/u,\omega\alpha_2 z/\beta_1 u\end{matrix};\frac{\omega\alpha_1}{\beta_1}\right]. \tag{30}
$$

Clearly, we may change β_2 to β_1 with $u \to \gamma_1$, and obtain a different transformation formula. If we let $z = \omega$ in (30), then it is easily seen that the ${}_3\Phi_2$ on the right-hand side of the equation becomes a ${}_2\Phi_1$. As the three ${}_2\Phi_1$ are summable, the cyclic hypergeometric series ${}_3\Phi_2$ is also summable for argument $z = \omega$. We conclude:

Theorem 2: Every cyclic basic hypergeometric series ${}_3\Phi_2$ has a transformation formula given by (30), and is summable for $z = \omega$.

2.4. *The Saalschützian ${}_4\Phi_3$ and the Star-Triangle Relation*

2.4.1. *Summation Formula*

Consider a Saalschützian ${}_4\Phi_3$ for argument $z = \omega$, and use the convolution theorem to express it as

$$
{}_4\Phi_3\left[\begin{matrix}\omega,\omega^a\alpha_1,\omega^b\alpha_2,\omega^c\alpha_3\\ \omega^a\beta_1,\omega^b\beta_2,\omega^c\beta_3\end{matrix};\omega\right] =
$$
$$
N^{-1}\sum_{k=0}^{N-1}{}_3\Phi_2\left[\begin{matrix}\omega,\omega^a\alpha_1,\omega^b\alpha_2\\ \omega^a\beta_1,\omega^b\beta_2\end{matrix};\omega^k\gamma\right]{}_2\Phi_1\left[\begin{matrix}\omega,\omega^c\alpha_3\\ \omega^c\beta_3\end{matrix};\frac{\omega^{1-k}}{\gamma}\right], \tag{31}
$$

where

$$
\gamma^N = \frac{(1-\alpha_3^N)}{(1-\beta_3^N)} = \frac{(1-\beta_1^N)(1-\beta_2^N)}{(1-\alpha_1^N)(1-\beta_2^N)} = \gamma_1^N\gamma_2^N. \tag{32}
$$

The first part of this equation and the Saalschütz condition $\omega^2\alpha_1\alpha_2\alpha_3 = \beta_1\beta_2\beta_3$, may be used to solve α_3 and also β_3. This gives

$$
\alpha_3^N = \frac{1-\gamma^N}{1-(\alpha_1\alpha_2\gamma/\beta_1\beta_2)^N}, \qquad \beta_3 = \frac{\omega^2\alpha_1\alpha_2\alpha_3}{\beta_1\beta_2}. \tag{33}
$$

Now we use the transformation formula (30) for ${}_3\Phi_2$ thrice, i.e.

$$
\begin{aligned}
{}_3\Phi_2\left[\begin{matrix}\omega,\omega^a\alpha_1,\omega^b\alpha_2\\ \omega^a\beta_1,\omega^b\beta_2\end{matrix};\gamma\omega^k\right] &= \hat{A}\,{}_3\Phi_2\left[\begin{matrix}\omega,\omega^k\gamma/u,\omega^{b-a}\bar{\beta}_3^*/u\\ \omega/u,\omega^{+k+b-a}\bar{\alpha}_3^*\gamma/u\end{matrix};\frac{\omega\alpha_1}{\beta_1}\right]\\
&= \hat{A}\hat{B}\,{}_3\Phi_2\left[\begin{matrix}\omega,\ \bar{\gamma}_1^*,\ \omega^k\bar{\gamma}_2\\ \omega/\bar{\gamma}_1,\omega^{k+1}/\bar{\gamma}_2^*\end{matrix};\frac{\omega^b\beta_2}{\omega^a\alpha_1}\right]\\
&= \hat{A}\hat{B}\hat{C}\,{}_3\Phi_2\left[\begin{matrix}\omega,\omega^{b-a}\bar{\alpha}_2^*,\bar{\alpha}_1^*\\ \omega^{b-a}\bar{\beta}_2^*,\bar{\beta}_1^*\end{matrix};\frac{\omega^{1-k}}{\gamma}\right],
\end{aligned} \tag{34}
$$

where the first line is identical to (30) in which

$$
\begin{cases}\bar{\alpha}_1 = \omega\alpha_2/\beta_3,\\ \bar{\beta}_1 = \beta_2/\alpha_3,\end{cases}\quad
\begin{cases}\bar{\alpha}_2 = \omega\alpha_1/\beta_3,\\ \bar{\beta}_2 = \beta_1/\alpha_3,\end{cases}\quad
\begin{cases}\bar{\alpha}_3 = \omega/\bar{\beta}_3^* = \omega\alpha_1/\beta_2,\\ \bar{\beta}_3 = \omega/\bar{\alpha}_3^* = \beta_1/\alpha_2,\end{cases} \tag{35}
$$

with superscript $*$ denoting the arrow-reversing operation described in (18), and

$$\hat{A} = N^{-1}\ {}_2\Phi_1\left[\begin{matrix}\omega,\omega^a\alpha_1\\ \omega^b\beta_2\end{matrix};u\right]{}_2\Phi_1\left[\begin{matrix}\omega,\omega^b\alpha_2\\ \omega^a\beta_1\end{matrix};\frac{\omega^k\gamma}{u}\right]. \tag{36}$$

If we denote

$$\gamma_j^N = \frac{1-\beta_j^N}{1-\alpha_j^N}, \qquad \bar{\gamma}_j^N = \frac{\bar{\beta}_j^N}{(\bar{\gamma}_j^*\bar{\alpha}_j)^N} = \frac{1-\bar{\beta}_j^N}{1-\bar{\alpha}_j^N}, \tag{37}$$

then by using (35) and (33), it is easy to verify that

$$\bar{\gamma}_1 = \gamma_2/\bar{\gamma}_3^*, \qquad \bar{\gamma}_2 = \gamma_1/\bar{\gamma}_3, \tag{38}$$

which in turn can be used to show

$$\bar{\gamma}_1\bar{\gamma}_2 = \gamma/\bar{\gamma}_3\bar{\gamma}_3^* = \gamma\beta_3/\omega\alpha_3 \quad \text{leading to} \quad \gamma_3 = \omega/\gamma = \bar{\gamma}_1^*\bar{\gamma}_2^*. \tag{39}$$

Using (29) in which $z = \gamma$, it is easy to verify that

$$\frac{1-(\bar{\alpha}_3^*\gamma/u)^N}{1-(\bar{\beta}_3^*/u)^N} = \left(\frac{\gamma_2}{\bar{\gamma}_3^*}\right)^N = \bar{\gamma}_1^N, \quad \frac{1-(1/u)^N}{1-(\gamma/u)^N} = \left(\frac{\bar{\gamma}_3^*\alpha_1}{\gamma_2\beta_1}\right)^N = (\bar{\gamma}_1^*)^N. \tag{40}$$

From (18) and (35), it follows that $\bar{\gamma}_1\bar{\gamma}_1^* = \omega\alpha_1/\beta_1$. As a consequence, we may use the transformation formula (30) again to obtain the second equality in (34) with

$$\hat{B} = N^{-1}\ {}_2\Phi_1\left[\begin{matrix}\omega,\omega^k\gamma/u\\ \omega/u\end{matrix};\bar{\gamma}_1^*\right]{}_2\Phi_1\left[\begin{matrix}\omega,\omega^{b-a}\bar{\beta}_3^*/u\\ \omega^{k+b-a}\bar{\alpha}_3^*\gamma/u\end{matrix};\bar{\gamma}_1\right]. \tag{41}$$

Furthermore, from (37), we find that

$$\frac{1-(1/\bar{\gamma}_j)^N}{1-(\bar{\gamma}_j^*)^N} = \bar{\alpha}_j^N, \qquad \frac{1-(1/\bar{\gamma}_j^*)^N}{1-(\bar{\gamma}_j)^N} = (\bar{\alpha}_j^*)^N, \tag{42}$$

while from (35) and (18) we obtain $\bar{\alpha}_1\bar{\alpha}_2^* = \beta_2/\alpha_1$. Using the transformation formula (30) for the third time, we arrive at (34) with

$$\hat{C} = N^{-1}\ {}_2\Phi_1\left[\begin{matrix}\omega,\bar{\gamma}_1^*\\ \omega/\bar{\gamma}_1\end{matrix};\bar{\alpha}_1\right]{}_2\Phi_1\left[\begin{matrix}\omega,\ \omega^k\bar{\gamma}_2\\ \omega^{k+1}/\bar{\gamma}_2^*\end{matrix};\omega^{b-a}\bar{\alpha}_2^*\right]. \tag{43}$$

Denoting

$$W_i(n) = \gamma_i^n\frac{(\alpha_i;\omega)_n}{(\beta_i;\omega)_n}, \qquad \overline{W}_i(n) = \overline{W}_i^*(-n) = \bar{\gamma}_i^n\frac{(\bar{\alpha}_i;\omega)_n}{(\bar{\beta}_i;\omega)_n} \tag{44}$$

and using recursion formula (20) for ${}_2\Phi_1$ in these constants $\hat{A}$, $\hat{B}$ and $\hat{C}$, we find

$$\begin{aligned}{}_3\Phi_2\left[\begin{matrix}\omega,\omega^a\alpha_1,\omega^b\alpha_2\\ \omega^a\beta_1,\omega^b\beta_2\end{matrix};\omega^k\gamma\right] &= \frac{\overline{W}_3^*(b-a)\overline{W}_2^*(b-a)}{\overline{W}_1(a)\overline{W}_2(b)}\\ &\times \frac{\omega^{-kb}D\alpha_3^{-k}(\gamma;\omega)_k}{(\gamma\beta_3/\omega\alpha_3;\omega)_k}{}_3\Phi_2\left[\begin{matrix}\omega,\omega^{b-a}\bar{\alpha}_2^*,\bar{\alpha}_1^*\\ \omega^{b-a}\bar{\beta}_2^*,\bar{\beta}_1^*\end{matrix};\frac{\omega^{1-k}}{\gamma}\right],\end{aligned} \tag{45}$$

where $D = [\hat{A}\hat{B}\hat{C}]_0$ with $a = b = c = k = 0$ in (36), (41) and (43). Next, the recursion formula (20) is used to write

$${}_2\Phi_1\left[\begin{matrix}\omega,\omega^c\alpha_3\\ \omega^c\beta_3\end{matrix};\frac{\omega^{1-k}}{\gamma}\right] = \frac{R\omega^{ck}(\gamma\beta_3/\omega\alpha_3;\omega)_k}{DW_3(c)\alpha_3^{-k}(\gamma;\omega)_k}, \quad \frac{R}{D} = {}_2\Phi_1\left[\begin{matrix}\omega,\alpha_3\\ \beta_3\end{matrix};.\right]\frac{\omega}{\gamma}. \tag{46}$$

Substituting these two equations into the convolution formula (31) and using

$$\frac{W(n+a)}{W(a)} = \gamma^n \frac{(\omega^a\alpha;\omega)_n}{(\omega^a\beta_i;\omega)_n}, \tag{47}$$

we find

$$W_1(a)W_2(b)W_1(c)\,{}_4\Phi_3\left[\begin{matrix}\omega, \omega^a\alpha_1, \omega^b\alpha_2, \omega^c\alpha_3 \\ \omega^a\beta_1, \omega^b\beta_2, \omega^c\beta_3\end{matrix};\omega\right] =$$
$$N^{-1}R\overline{W}_3(a-b)\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\omega^{k(c-b-l)}\overline{W}_2(a-b-l)\overline{W}_1(-l), \tag{48}$$

where the summation over k can be carried out resulting in the delta function $N\delta_{l,c-b}$. This then proves the following theorem:

Theorem 3: Every cyclic Saalschützian basic hypergeometric series ${}_4\Phi_3$ is summable for $z = \omega$.

2.4.2. *Star-Triangle Relation*

Furthermore, (48) is also the star-triangle equation

$$\sum_{d=0}^{N-1} W_1(a-d)W_2(b-d)W_3(c-d) = R\,\overline{W}_3(a-b)\overline{W}_2(a-c)\overline{W}_1(b-c), \tag{49}$$

shown in Fig. 2. The weights W and $\overline{W}$ are defined in (44), in which the parameters α_i, β_i and $\bar{\alpha}_i$, $\bar{\beta}_i$ are related by (35), while γ_i and $\bar{\gamma}_i$ are related by (38) and (39). These relations are very symmetric.

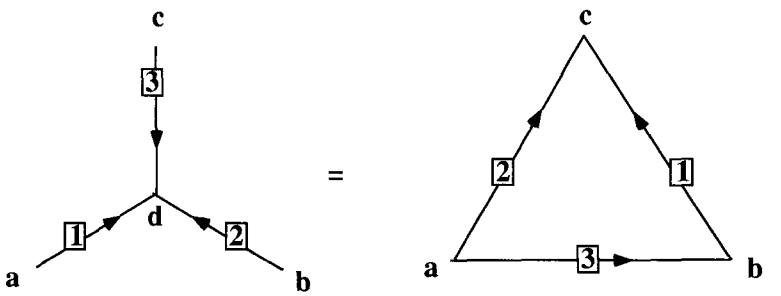

Fig. 2. Star-Triangle Relation, with $\gamma_1 = \bar{\gamma}_2\bar{\gamma}_3$, $\gamma_2 = \bar{\gamma}_1\bar{\gamma}_3^*$ and $\gamma_3 = \bar{\gamma}_1^*\bar{\gamma}_2^*$.

2.4.3. *The Constant R*

The constant R was originally given in Ref. 3. A proof was published in Ref. 16 and their proof can also be used here. By defining the matrices

$$(\mathcal{A}_2)_{b,d} = W_2(b-d), \quad (\mathcal{A}_3)_{c,d} = W_3(c-d), \quad (\mathcal{A}_1^a)_{d,d'} = \delta_{d,d'}W_1(a-d),$$

$$(\bar{\mathcal{A}}_1)_{b,c}=\overline{W}_1(b-c), \quad (\bar{\mathcal{A}}_2^a)_{c,c'}=\delta_{c,c'}\overline{W}_2(a-c), \quad (\bar{\mathcal{A}}_3^a)_{b,b'}=\delta_{b,b'}\overline{W}_2(a-b), \tag{50}$$

the star-triangle equation (49) may be expressed as

$$(\mathcal{A}_2\mathcal{A}_1^a\mathcal{A}_3^T)_{b,c}=R(\bar{\mathcal{A}}_3^a\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2^a)_{b,c} \quad \text{or} \quad \mathcal{A}_2\mathcal{A}_1^a\mathcal{A}_3^T=R\bar{\mathcal{A}}_3^a\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2^a. \tag{51}$$

The determinants are also equal, i.e.

$$R^N=\prod_{l=0}^{N-1}\frac{W_1(l)}{\overline{W}_2(l)\overline{W}_3(l)}\frac{\det\mathcal{A}_2\det\mathcal{A}_3}{\det\bar{\mathcal{A}}_1} \tag{52}$$

This gives the constant R in terms of determinants of matrices $\mathcal{A}$ defined in (50), which can be evaluated by Baxter's formula (23). Alternatively, R is seen from (46) to be a product of seven ${}_2\Phi_1$. It can also be evaluated using (26), which is much more tedious, and after many cancellations, this yields the same result. We have thus avoided the complexity in the Riemann surface by relegating it to the multiplicative constant R in the star-triangle equation.

2.4.4. *Rapidity Lines*

To form commuting transfer matrices, it is necessary to assign rapidity lines to the weights. There are two possible weights shown in Fig. 1. There are two essentially different choices for the directions of the arrows.

Original Choice

By assigning the rapidity lines as we originally did in Ref. 3, also shown in Fig. 3, then it is easily seen from Figs. 1 and 3 that

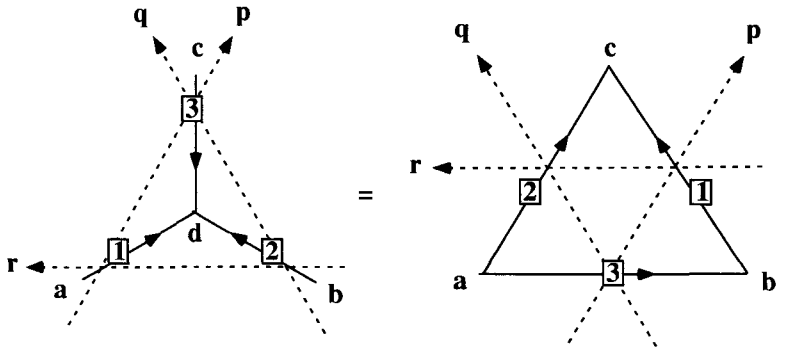

Fig. 3. Original choice of the rapidities.

$$\begin{cases} W_1(n)=W_{pr}(n), \\ \overline{W}_1(n)=\overline{W}_{pr}(n), \end{cases} \quad \begin{cases} W_2(n)=\overline{W}_{qr}(n), \\ \overline{W}_2(n)=W_{qr}(n), \end{cases} \quad \begin{cases} W_3(n)=\overline{W}^*_{pq}(n), \\ \overline{W}_3(n)=W_{pq}(n). \end{cases} \tag{53}$$

This means

$$\begin{cases} \alpha_1 = \alpha_{pr}, \\ \beta_1 = \beta_{pr}, \\ \gamma_1 = \gamma_{pr}, \end{cases} \quad \begin{cases} \alpha_2 = \bar{\alpha}_{qr}, \\ \beta_2 = \bar{\beta}_{qr}, \\ \gamma_2 = \bar{\gamma}_{qr}, \end{cases} \quad \begin{cases} \alpha_3 = \omega/\bar{\beta}_{pq}, \\ \beta_3 = \omega/\bar{\alpha}_{pq}, \\ \gamma_3 = \gamma^*_{pq}, \end{cases}$$

$$\begin{cases} \bar{\alpha}_1 = \bar{\alpha}_{pr}, \\ \bar{\beta}_1 = \bar{\beta}_{pr}, \\ \bar{\gamma}_1 = \bar{\gamma}_{pr}, \end{cases} \quad \begin{cases} \bar{\alpha}_2 = \alpha_{qr}, \\ \bar{\beta}_2 = \beta_{qr}, \\ \bar{\gamma}_2 = \gamma_{qr}, \end{cases} \quad \begin{cases} \bar{\alpha}_3 = \alpha_{pq}, \\ \bar{\beta}_3 = \beta_{pq}, \\ \bar{\gamma}_3 = \gamma_{pq}. \end{cases} \tag{54}$$

Consequently, we find from (35)

$$\begin{cases} \bar{\alpha}_{pr} = \bar{\alpha}_{qr}\bar{\alpha}_{pq}, \\ \bar{\beta}_{pr} = \bar{\beta}_{qr}\bar{\beta}_{pq}/\omega, \end{cases} \quad \begin{cases} \alpha_{qr} = \alpha_{pr}\bar{\alpha}_{pq}, \\ \beta_{qr} = \beta_{pr}\bar{\beta}_{pq}/\omega, \end{cases} \quad \begin{cases} \alpha_{pq} = \omega\alpha_{pr}/\bar{\beta}_{qr}, \\ \beta_{pq} = \beta_{pr}/\bar{\alpha}_{qr}. \end{cases} \tag{55}$$

From the relations for $\bar{\alpha}_{pr}$ and $\bar{\beta}_{pr}$, we see that we would like to have the products $\bar{\alpha}_{qr}\bar{\alpha}_{pq}$ and $\bar{\beta}_{qr}\bar{\beta}_{pq}$ independent of q. For this to happen, we must have $\bar{\alpha}_{qr}$ and $\bar{\alpha}_{pq}$ containing the same q-dependent factor, say x_q, one in the denominator, the other in the numerator, such that the dependence on q cancels out upon multiplication. A similar reasoning holds for $\bar{\beta}_{pq}$. In fact, we find the only choices are

$$\bar{\alpha}_{pq} = x_q/x_p, \quad \bar{\beta}_{pq} = \omega y_p/y_q. \tag{56}$$

Using this in the second and third brackets of (55), we find

$$\alpha_{pq} = \omega x_p/y_q, \quad \beta_{pq} = \omega x_q/y_p. \tag{57}$$

It is easily verified that these choices satisfy the Saalschütz condition in (17). The periodicity requirement on the argument at $z = \omega$

$$\omega = \gamma_1\gamma_2\gamma_3 = \prod_{j=1}^{3} \left[(1-\beta_j^N)/(1-\alpha_j^N)\right]^{1/N} \tag{58}$$

can be satisfied, if

$$x_s^N + y_s^N = k(1 + x_s^N y_s^N), \qquad s = p, q, r. \tag{59}$$

Solving this equation for x_s and substituting the solution into (37), we find

$$\gamma_{pq} = \mu_p y_q/\mu_q y_p \quad \bar{\gamma}_{pq} = \omega\mu_p x_p \mu_q/y_q, \quad \mu_s = (1 - kx_s^N)/k'. \tag{60}$$

This reproduces exactly the integrable solution found earlier.[3]

Other Distinct Choice

Only by flipping the directions of the middle rapidity line q, do we find a distinct arrangement of weights. This results in the equation

$$\sum_{d=0}^{N-1} W_{pr}(a-d)W_{rq}(b-d)W_{qp}(c-d) = R\,\overline{W}_{qp}(a-b)\overline{W}_{rq}(c-a)\overline{W}_{pr}(b-c). \tag{61}$$

Flipping other lines merely gives permutations of these rapidity lines in the two star-triangle equations, as can be seen from Fig. 1 and Fig. 4. To have the relation

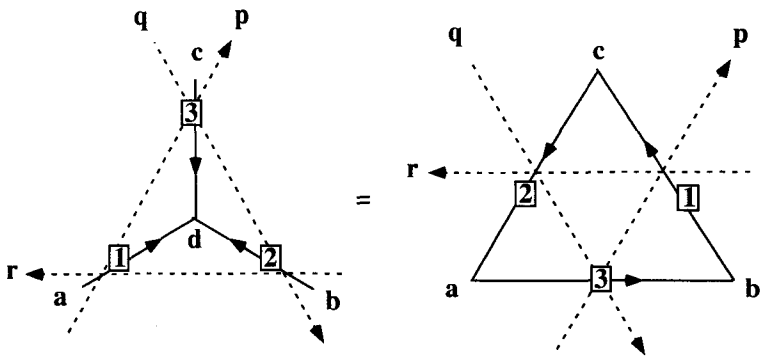

Fig. 4. Flipping the arrow of q.

(35) to hold, we must have

$$\bar{\alpha}_{pr} = \omega\alpha_{rq}/\beta_{qp}, \qquad \bar{\beta}_{pr} = \beta_{rq}/\alpha_{qp}, \tag{62}$$

for which to be satisfied, we must choose

$$\alpha_{pr} = \rho f_p g_r, \qquad \beta_{pr} = g_p f_r. \tag{63}$$

The Saalschütz condition in (17) yields $\omega^2\rho^3 = 1$. It is then easy to verify that it is not possible to find ρ, f_s and g_s satisfying condition (58). Since the relation (61) is more symmetric than (49) when comparing Fig. 3 with Fig. 4, the extra symmetry requirement on the weights makes a solution impossible in the present case.

3. The $N \to \infty$ Limits

3.1. *Star-Triangle Equation as a Double-Sided Hypergeometric Identity*

In the limit $N \to \infty$, with $\alpha_i = \omega^{a_i}$ and $\beta_i = \omega^{b_i}$, and allowing the spin n in (44) and thus the summation index d in (49) to run through all integers, we find[12,13] that if the Saalschützian condition

$$a_1 + a_2 + a_3 + 2 = b_1 + b_2 + b_3. \tag{64}$$

and condition resulting from (58)

$$\sin\pi a_1 \sin\pi a_2 \sin\pi a_3 = \sin\pi b_1 \sin\pi b_2 \sin\pi b_3, \tag{65}$$

are satisfied, the star-triangle equation (49) becomes

$$\sum_{n=-\infty}^{\infty} \frac{(a_1)_{m_1+n}(a_2)_{m_2+n}(a_3)_{m_3+n}}{(b_1)_{m_1+n}(b_2)_{m_2+n}(b_3)_{m_3+n}} = R_\infty \frac{(\bar{a}_1)_{m_1-m_2}(\bar{a}_2)_{m_2-m_3}(\bar{a}_3)_{m_1-m_3}}{(\bar{b}_1)_{m_1-m_2}(\bar{b}_2)_{m_2-m_3}(\bar{b}_3)_{m_1-m_3}}, \tag{66}$$

where

$$\begin{cases} \bar{a}_1 = 1 + a_2 - b_3, \\ \bar{b}_1 = b_2 - a_3, \end{cases} \quad \begin{cases} \bar{a}_2 = 1 + a_1 - b_3, \\ \bar{b}_2 = b_1 - a_3, \end{cases} \quad \begin{cases} \bar{a}_3 = 1 + a_1 - b_2, \\ \bar{b}_3 = b_1 - a_2. \end{cases} \tag{67}$$

3.1.1. *Double-Sided Hypergeometric Identity*

The equation (66) may be rewritten as the double-sided summation identity[13]

$$\sum_{n=-\infty}^{\infty} \prod_{i=1}^{3} \frac{\Gamma(a_i+n)}{\Gamma(b_i+n)} = \frac{G(a_1,a_2,a_3|b_1,b_2,b_3)}{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma(b_i-a_j)}, \tag{68}$$

where

$$\begin{aligned} G(a_1,a_2,a_3|b_1,b_2,b_3) &= \frac{\pi^5}{\sin\pi a_2 \sin\pi a_3 \prod_{i=1}^{3}\sin\pi(b_i-a_1)} \\ &= \prod_{j=2}^{3}\Gamma(a_j)\Gamma(1-a_j)\prod_{i=1}^{3}\Gamma(b_i-a_1)\Gamma(1-b_i+a_1), \end{aligned} \tag{69}$$

provided the two conditions (64) and (65) are satisfied. If we let $a_i \to a_i + m$ and $b_j \to a_j + m$, then these two conditions are still satisfied. This shows that the above two-sided identity holds for infinitely many different values of a_i and b_j and is rather unusual.

3.2. *Its Dual (Fourier Transform)*

If, instead of demanding that the spin values n in (44) and d in (49) remain integers, we let $d, n, N \to \infty$, while keeping the ratios $y = 2\pi n/N$ and $x = 2\pi d/N$ finite, we find that the summation over N values in (49) becomes an integral over the interval $[0, 2\pi]$. More specifically, we find the weight (18) to become[13]

$$W(a,b,x) = \Big(\frac{\sin\pi b}{\sin\pi a}\Big)^{(\frac{x}{2\pi} - \lfloor\frac{x}{2\pi}\rfloor)} \big|\sin\tfrac{1}{2}x\big|^{a-b}. \tag{70}$$

in this limit. For a_i and b_i satisfying the two conditions (64) and (65), we let

$$W_i(x) = W(a_i, b_i, x), \quad \overline{W}_i(x) = W(\bar{a}_i, \bar{b}_i, x), \tag{71}$$

and the star-triangle relation (49) becomes[13]

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}w\; W_1(x-w)\, W_2(y-w)\, W_3(z-w) \\ &\qquad = R_\infty\, \overline{W}_3(x-y)\, \overline{W}_1(y-z)\, \overline{W}_2(x-z). \end{aligned} \tag{72}$$

Since the weights are chiral, namely, $W(-x) \neq W(x)$, it is not possible to have both the weights and their Fourier transforms real. Thus the Fourier transform of (72) is an identity similar to (66), but not identical, and vice versa.

3.3. *Open Problems*

Finally, the weights in Sections 3.1 and 3.2 define integrable models, which are limiting cases of the original chiral Potts model, and are chiral extensions of the models in the works of Fateev and Zamolodchikov.[19,20] Since there are sets of N functional relations for the chiral Potts models, we expect there may then be infinitely many such functional relations for these models, and perhaps some more physical quantities in these ∞-state models can be evaluated.

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HAPPER'S CURIOUS DEGENERACIES AND YANGIAN

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We find raising and lowering operators distinguishing the degenerate states for the Hamiltonian $H = x(K + \frac{1}{2})S_z + \mathbf{K} \cdot \mathbf{S}$ at $x = \pm 1$ for spin 1 that was given by Happer et al.[1,2] to interpret the curious degeneracies of the Zeeman effect for condensed vapor of ^{87}Rb. The operators obey Yangian commutation relations. We show that the curious degeneracies seem to verify the Yangian algebraic structure for quantum tensor space and are consistent with the representation theory of $Y(sl(2))$.

1. Indecomposible Quantum Tensor Space

In Quantum Mechanics, a state is described in terms of wave function, i.e. $|\psi>$ is a vector in Hilbert space. If two particles described by $|\psi_{12}>$ are entangled, there should be "overlapping effect" between V_1 and V_2, i.e., besides V_1 and V_2 we should deal with $V_1 \otimes V_2$, the quantum tensor space. The simplest example is Breit-Rabi's Hamiltonian:

$$H_{\mathrm{BR}} = \mathbf{K} \cdot \mathbf{s} + xks_3, \tag{1.1}$$

where $\mathbf{s}$ and $\mathbf{K}$ stand for the spins of electron and atomic nucleus, respectively. $\mathbf{K}^2 = K(K+1)$. On account of the conservation of $\mathbf{K}^2$ and $m = K_3 + s_3$ two independent states are introduced:

$$\alpha_1 >= |K, m - \frac{1}{2} > |\frac{1}{2}, \frac{1}{2}>, \quad \alpha_2 >= |K, m + \frac{1}{2} > |\frac{1}{2}, -\frac{1}{2}> . \tag{1.2}$$

For a fixed m with the basis $\Phi = \begin{pmatrix} |\alpha_1> \\ |\alpha_2> \end{pmatrix}$, we have

$$H_{\rm BR}^{(m)} = -\frac{1}{4} + \frac{1}{2}[(xk+m)\sigma_3 + \sqrt{k^2-m^2}\sigma_1], \tag{1.3}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices. Eq. (1.3) can be diagonalized through a rotation:[1]

$$U(\varphi_m)H_{\rm BR}^{(m)}U(\varphi_m)^{-1} = H_{\rm BR}^{(m)}(\varphi_m), \quad \Phi^{(m)}(\varphi_m) = U(\varphi_m)\Phi^{(m)}, \tag{1.4}$$

where

$$\Phi^{(m)}(\varphi_m) = \begin{pmatrix} (\cos\frac{\varphi_m}{2})|\alpha_1> -(\sin\frac{\varphi_m}{2})|\alpha_2> \\ (\sin\frac{\varphi_m}{2})|\alpha_1> +(\cos\frac{\varphi_m}{2})|\alpha_2> \end{pmatrix}, \tag{1.5}$$

$$E = -\frac{1}{4} - \omega_m\sigma_3, \tag{1.6}$$

and

$$\cos\varphi_m = \frac{(xk+m)}{\omega_m}, \quad \omega_m^2 = (1+x^2)k^2 + 2xmk. \tag{1.7}$$

Noting that the rotation angle φ_m is m-dependent and m here cannot be replaced by the operator $K_3 + s_3$. This is because of the nonlinearity in m, i.e., the rotation should depend on the history. Observing Eq. (1.6) and Eq. (1.7) there is not degeneracy for the energy E, because the vanishing ω_m means a complex magnetic field.

However, there appears degeneracies for spin-1 in the experiment.[2] Why the Zeeman effect vanishes at the particular value of applied field? This is the main subject concerned in this paper.

2. Introduction of the Curious Degeneracies

The curious degeneracies observed in the experiment for condensed vapor of ^{87}Rb and ^{85}Rb[1] at 220° under pressure and applied magnetic field $B \sim 1500$ Gauss are converted into "anti-level-crossing" for the triplet $(S = 1)$.[1,2] To describe the Hamiltonian of a triplet dimer neglecting the quadrapole interaction, Happer et al. introduced[1,2]

$$H = \mathbf{K}\cdot\mathbf{S} + x(K + \frac{1}{2})S_z, \tag{2.1}$$

and pointed out that when $x = 1$ there appear the curious degeneracies for $S = 1$, where $\mathbf{K}$ and $\mathbf{S}$ are angular momentum and spin, respectively, $\mathbf{K}^2 = K(K+1)$ and $\mathbf{S}^2 = S(S+1)$ with $S = 1$. In Ref. 1, the eigenvectors corresponding to $E = -\frac{1}{2}$ had been given and an elegant discussion was made. However, there remain the following essential questions:

- Why the curious degeneracies occur only for $S = 1$?

• How to distinguish the degenerate states?

We would like to present the answer in this paper.

For $x = \pm 1$, the eigenequation

$$H\Psi_m = E_m \Psi_m \tag{2.2}$$

has three types of solutions whose eigenstates are denoted by α_T, α_D and α_B with the corresponding energies $E_T > E_D > E_B$, respectively. For the D-set, $H\alpha_{Dm} = -\frac{1}{2}\alpha_{Dm}$, there appear the curious degeneracies called Happer degeneracies that has been supported by the experiment.[2] The results of Happer can be summarized in the Table 1 ($\mathbf{G} = \mathbf{K} + \mathbf{S}$, $G_3 = m$).

	$G = K+1$	$G = K$	$G = K-1$		$D-$set	$T-$set	$B-$set
$K+1$	$--$			$\rightarrow$		$\alpha_{T,m=K+1}$	
K	$--$	$--$		$\rightarrow$	$\alpha_{D,m=K}$	$\alpha_{T,m=K}$	
$K-1$	$--$	$--$	$--$	$\rightarrow$	$\alpha_{D,m=K-1}$	$\alpha_{T,m=K-1}$	$\alpha_{B,m=K-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$--$	$--$	$--$	$\rightarrow$	α_{Dm}	α_{Tm}	α_{Bm}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-K+1$	$--$	$--$	$--$	$\rightarrow$	$\alpha_{D,m=-K+1}$	$\alpha_{T,m=-K+1}$	$\alpha_{B,m=-K+1}$
$-K$	$--$	$--$		$\rightarrow$		$\alpha_{T,m=-K}$	$\alpha_{B,m=-K}$
$-K-1$	$--$			$\rightarrow$	$\alpha_{D,m=-K-1}$		

Table 1

We emphasize that the states with $m = K+1$ and $m = -K$ for $x = 1$ ($m = -K-1$ and $m = K$ for $x = -1$) in the D-set are excluded. For simplicity we discuss the case for $x = 1$ henceforth. The eigenstates of H are linear combinations of the states of $G = K+1, K$ and $K-1$. Since the shortage of states with $m = K+1$ and $m = -K$ it is not surprise to appear the unusual thing to distinguish the m-dependent states, for example in Eq. (1.6).

3. Yangian as the Raising and Lowering Operator for the Degenerate States

Let us first recall how to establish the Lie algebraic structure in Quantum Mechanics. For the given $(2K+1)$ states denoted by $|K, K_3 = K>$, $|K, K_3 = K-1>, \cdots$, and $|K, K_3 = -K>$, the raising (or lowering) operator K_+ (or K_-) can be introduced such that for any $m = K_3$,

$$K_\pm |K, m> \sim |K, m \pm 1>, \tag{3.1}$$

and $K \pm |K, \pm K> = 0$. Through checking the commutation relations for $K_\pm$ and K_3, we say that the Lie algebraic structure is found if the commutation relations are closed. It is emphasized that there is not m-dependence in the operators $K_\pm$ in Eq.

(3.1), because the eigenvalues of K_3 are uniform. However, suppose the eigenvalues are not uniform, the raising and lowering operators should depend on m, i.e., it should indicate on which state the operators act. Actually, such "starting state" dependence occurs more often in nonlinear models.[3]

After calculations, we have found the raising operator for the D-set in the table 1 (at $x = \pm 1$):

$$J_+ = (m + K + 1)G_+ + j_+(a, b), \tag{3.2}$$

where

$$j_+(a, b) = aS_+ + bK_+ + \frac{1}{2}(S_3K_+ - S_+K_3), \tag{3.3}$$

and

$$a = -\frac{K}{2}, b - a = \frac{1}{2}(K + 1), G_+ = K_+ + S_+. \tag{3.4}$$

Noting that $(b - a)$ is independent of m. Whereas

$$J_- = -(m + K)G_- + j'_-(c, d), \tag{3.5}$$

where

$$j'_-(c, d) = cS_- + dK_- - \frac{1}{2}(S_3K_- - S_-K_3), \tag{3.6}$$

and

$$c = \frac{K}{2} + \frac{1}{2}, d - c = -\frac{K}{2}, G_- = K_- + S_-. \tag{3.7}$$

It can be checked that for $x = 1$, $J_+|\alpha_{D,m=K} >= 0$ and $J_+|\alpha_{D,m=-K-1} >= 0$.

Obviously the $J_\pm$ shown in Eq. (3.2) and Eq. (3.5) are special form of the Yangian operator:

$$\mathbf{J} = \lambda\mathbf{G} + \mathbf{j}, \tag{3.8}$$

where

$$\mathbf{j} = \mu\mathbf{K} + \gamma\mathbf{S} - \frac{i}{2}\mathbf{S} \times \mathbf{K}, \tag{3.9}$$

and λ, μ, γ are arbitrary constants. A set formed by both $\mathbf{J}$ and $\mathbf{j}$ satisfy $Y(sl(2))$ defined by Drinfeld,[4] and is related to the Yang-Baxter equations.[5,6]

4. Yangian Algebra

The commutation relations for $\mathbf{J}$ and the total angular momentum $\mathbf{I} = \mathbf{G} = \mathbf{S} + \mathbf{K}$ form the so-called Yangian algebra associated with $sl(2)$. The parameters μ and γ play the important role in the representation theory of Yangian given by Chari and Pressley.[7] Many chain models possess the Yangian symmetry, for example, for 1-d Hubbard model and Haldane-Shastry model.[8] The set $\{\mathbf{I}, \mathbf{J}\} = Y(sl(2))$ obeys the commutation relations of $Y(sl(2))$ ($A_\pm = A_1 \pm \sqrt{-1}A_2$):

$$[I_3, I_\pm] = \pm I_\pm, \quad [I_+, I_-] = 2I_3, \quad (sl(2)); \tag{4.1}$$

$$[I_3,\ J_\pm] = [J_3,\ I_\pm] = \pm J_\pm,\ \ [I_+,\ J_-] = [J_+,\ I_-] = 2J_3, \tag{4.2}$$

(i.e. $[I_i,\ J_j] = \sqrt{-1}\varepsilon_{ijk}J_k$) and nonlinear relation

$$[J_3,\ [J_+,\ J_-]] = \frac{1}{4}I_3(I_+J_- - J_+I_-) \tag{4.3}$$

that forms an infinitely dimensional algebra. All the other relations given in Ref. 4 can be obtained from Eq. (4.1)–Eq. (4.3) together with the Jacobian identities.[9,10]

The essential difference between the representations of Yangian algebras and those of Lie algebras is the appearance of the free parameters μ and γ whose originally physical meaning is one-dimensional momentum. Their special choice specifies a particular model. Applying the Yangian representation theory to Hydrogen atom, it yields the correct spectrum ($\sim n^{-2}$) that is the simplest example of the application of Yangian in Quantum Mechanics.[10] Now the Happer's degeneracies can be viewed as another example. Furthermore, we would like to make the following remarks:

(a) The elements of J_+ given by Eq. (3.2)

$$< \alpha_{Dm'}|J_+|\alpha_{Dm} > \sim < \alpha_{Dm'}|K_+|\alpha_{Dm} > \neq 0,$$

because $< \alpha_{Dm'}|\mathbf{S}|\alpha_{Dm} > = < \alpha_{Dm'}|\mathbf{S}\times\mathbf{K}|\alpha_{Dm} > = 0$, as pointed out in Ref. 1 (see Eq. (2.23) in Ref. 1). This indicates that the role played by J_+ in the "D-direction" is like that played by K_+. Why do we need a Yangian? The terms of S_+ and $(\mathbf{K}\times\mathbf{S})_+$ should be added to guarantee $< \alpha_{Tm'}|J_+|\alpha_{Dm} > = < \alpha_{Bm'}|J_+|\alpha_{Dm} > = 0$, namely, if only acting K_+ on α_{Dm} it yields non-vanishing transitions to $\alpha_{Tm'}$ and $\alpha_{Bm'}$ that no longer preserves the D-set. The part other than K_+ in the Yangian J_+ given by Eq. (3.2) exactly cancel the nonvanishing contribution received from "T-" and "B-direction".

(b) Observing the process determining parameters a and b in Eq. (3.3), the reason for the existence of solution of a and b is clear. For $S = 1$, the eigenvector of H is formed by three base. Apart of an over-all normalization factor there are two independent coefficients. In requiring $J_+\alpha_{Dm} \sim \alpha_{Dm+1}$, we have to compare the coefficients of the independent base in $J_+\alpha_{Dm}$ and α_{Dm+1} to determine the unknown parameters a and b. For spin $S = 1$, there are just two equations for a and b. However, for spin $S > 1$, in general, one is unable to find solution for a and b to fit more than two equations. Therefore, the Yangian description of the curious degeneracies admits only $S = 1$ for arbitrary K. This is consistent with experiment.[1,2]

(c) In fact, the parameters appearing in J_+ and J_- exactly coincide with the conditions of the existence of the subrepresentations of the Yangian.[7] Following the theorem in Ref. 7, for $a - b = -\frac{K}{2} - \frac{1}{2}$ the subspace spanned by vectors with $G = K + 1$ is the unique irreducible subrepresentation of $Y(sl(2))$, that is, the states with $G = K + 1$ are stable under the action of $\mathbf{J}$. Note that the existence and uniqueness of subrepresentation is only related to the difference of a and b. Moreover, for the given a and b in Eq. (3.4), the action of J_+ on the states with

$G = K + 1$ is given by $J_+\alpha_{G=K+1,m} = (m + K + 1)G_+\alpha_{G=K+1,m}$ and at the same time, J_+ will make the states with $G = K$ and $G = K - 1$ transit to $G = K + 1$, but not vice versa, called "directional transition",[9] i.e. the transition given rise by Yangian goes in one way. Thus, for the given a and b in Eq. (3.4), the set of states with $G = K + 1$ and D-set are stable under the action of J_+ simultaneously. For $c - d = \frac{K}{2}$, $G = K - 1$ is the unique irreducible subrepresentation and for c and d given by Eq. (3.7), acting J_- on the states with $G = K - 1$, we have $J_-\alpha_{G=K-1,m} = -(m + K)G_-\alpha_{G=K-1,m}$. Therefore the representation theory of $Y(sl(2))$ tells that the relationship between $a - b$ and $c - d$ given by Eq. (3.4) and Eq. (3.7), respectively, should be held to preserve the states with $G = K + 1$ (or $G = K - 1$) that possesses Lie algebraic behavior.

(d) We have seen that the J_- is not the conjugate of J_+. Such a phenomenon is reasonable because α_{Dm} is neither the Lie-algebraic state nor symmetry of H. In fact, if α is not an eigenstate of $\mathbf{I}^2$ ($\mathbf{I}$ belongs to a Lie algebra) and $I_+\alpha \sim \alpha_1$, we cannot have $I_-\alpha_1 \sim \alpha$. Now there is the similarity for Yangian. Moreover, the D-set is not a subrepresentation of $Y(sl(2))$, i.e., D-set cannot be stable under all the actions of $\mathbf{J}$, but stable under J_+ and J_- with the different parameters which just satisfy the condition for subrepresentation of Yangian.

(e) The third component of $\mathbf{J}$ takes the form $J_3 = aS_z + bK_z + S_+K_- - S_-K_+$. For any parameters, the action of J_3 will not keep the D-set. But, with the suitable $a - b = 1$, the operator $J_3 + 2(2K + 1)S_z^2$ will keep the D-set.

(f) We emphasized that the m appearing in Eq. (3.2) and Eq. (3.5) cannot be replaced by the operator G_3. It appears as a parameter in Yangian. The m-dependents only indicates that the raising or lowering operation depends on "history" in difference from the Lie algebraic structure.

In conclusion we have read of a new type of algebra structure(Yangian) from the Happer's degeneracies and such an algebra had been ready by Drinfeld.[4] All the analysis coincides with the representation theory of $Y(sl(2))$[7] for the special choice of a, b in J_+ and c, d in J_-. It also leads to the fact that only $S = 1$ is allowed to yield the curious degeneracies. If the Zeeman effect tells Lie algebra, then the curious degeneracies possibly tell the existence of Yangian.

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FRACTIONAL STATISTICS IN SOME EXACTLY SOLVABLE MODELS WITH PT INVARIANT INTERACTIONS

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Here we review a method for constructing exact eigenvalues and eigenfunctions of a many-particle quantum system, which is obtained by adding some nonhermitian but PT invariant (i.e., combined parity and time reversal invariant) interaction to the Calogero model. It is shown that such extended Calogero model leads to a real spectrum obeying generalised exclusion statistics. It is also found that the corresponding exchange statistics parameter differs from the exclusion statistics parameter and exhibits a 'reflection symmetry' provided the strength of the PT invariant interaction exceeds a critical value.

1. Introduction

It is well known that integrable dynamical models and spin chains with long range interactions exhibit fractional statistics or generalised exclusion statistics (GES),[1] which is believed to play an important role in many strongly correlated systems of condensed matter physics. The A_{N-1} Calogero model (related to A_{N-1} Lie algebra) is the simplest example of such dynamical model, containing N particles on a line and with Hamiltonian given by Refs. 2,3

$$H = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{\omega^2}{2}\sum_{j=1}^{N}x_j^2 + \frac{g}{2}\sum_{j\neq k}\frac{1}{(x_j - x_k)^2}\,, \tag{1}$$

where g is the coupling constant associated with long-range interaction. One can exactly solve this Calogero model and find out the complete set of energy eigenvalues as

$$E_{n_1,n_2,\cdots,n_N} = \frac{N\omega}{2}\left[1 + (N-1)\nu\right] + \omega\sum_{j=1}^{N}n_j. \tag{2}$$

Here n_js are non-negative integer valued quantum numbers with $n_j \leq n_{j+1}$ and ν is a real positive parameter which is related to g as

$$g = \nu^2 - \nu\,. \tag{3}$$

It may be noted that, apart from a constant shift for all energy levels, the spectrum (2) coincides with that of N number of free bosonic oscillators. Furthermore, one can easily remove the above mentioned constant shift for all energy levels and express (2) exactly in the form of energy eigenvalues for free oscillators: $E_{n_1,n_2,\cdots,n_N} = \frac{N\omega}{2} + \omega \sum_{j=1}^{N} \bar{n}_j$, where $\bar{n}_j = n_j + \nu(j-1)$ are quasi-excitation numbers. However it is evident that these $\bar{n}_j$s are no longer integers and they satisfy a modified selection rule given by $\bar{n}_{j+1} - \bar{n}_j \geq \nu$, which restricts the difference between the quasi-excitation numbers to be at least ν apart. As a consequence, the Calogero model (1) provides a microscopic realisation for fractional statistics with ν representing the corresponding GES parameter.[4-7]

Recently, theoretical investigations on different nonhermitian Hamiltonians have received a major boost because many such systems, whenever they are invariant under combined parity and time reversal (PT) symmetry, lead to real energy eigenvalues.[8-11] This seems to suggest that the condition of hermiticity on a Hamiltonian can be replaced by the weaker condition of PT symmetry to ensure that the corresponding eigenvalues would be real ones. However, till now this is merely a conjecture supported by several examples. Moreover, in almost all of these examples, the Hamiltonians of only one particle in one space dimension have been considered. Therefore, it should be interesting to test this conjecture for the cases of nonhermitian N-particle Hamiltonians in one dimension which remain invariant under the PT transformation:[12]

$$i \to -i, \quad x_j \to -x_j, \quad p_j \to p_j\,, \tag{4}$$

where $j \in [1, 2, \cdots, N]$, and x_j ($p_j \equiv -i\frac{\partial}{\partial x_j}$) denotes the coordinate (momentum) operator of the j-th particle. In particular, one may construct an extension of Calogero model by adding to it some nonhermitian but PT invariant interaction, and enquire whether such extended model would lead to real spectrum.

The purpose of the present article is to review the progress[7,12] on the above mentioned problem for some special cases, where the PT invariant extension of Calogero model can be solved exactly. In Sec.2 of this article we consider such a PT invariant extension of A_{N-1} Calogero model and show that, within a certain range of the related parameters, this extended Calogero model yields real eigenvalues. Next, in Sec.3, we explore the connection of these real eigenvalues with fractional statistics. Section 4 is the concluding section.

2. Exact solution of an extended Calogero model

Let us consider a Hamiltonian of the form[7]

$$\mathcal{H} = H + \delta \sum_{j \neq k} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j}, \tag{5}$$

where H is given by eqn.(1) and δ is a real parameter. It may be observed that though the Hamiltonian (5) violates hermiticity property due to the presence of

momentum dependent term like $\delta \sum_{j \neq k} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j}$, it remains invariant under the combined PT transformation (4). Next we recall that, A_{N-1} and B_N Calogero models as well as their distinguishable variants have been solved recently by mapping them to a system of free oscillators.[13–16] With the aim of solving the PT invariant extension (5) of A_{N-1} Calogero model through a similar produce, we assume that (justification for this assumption will be given later) the ground state wave function for this extended model is given by

$$\psi_{gr} = e^{-\frac{\omega}{2} \sum_{j=1}^{N} x_j^2} \prod_{j<k} (x_j - x_k)^{\nu}, \tag{6}$$

where ν is a real positive number which is related to the coupling constants g and δ as

$$g = \nu^2 - \nu(1 + 2\delta)\,. \tag{7}$$

Now if we use the expression (6) for a similarity transformation to the Hamiltonian (5), it reduces to an 'effective Hamiltonian' of the form

$$\mathcal{H}' = \psi_{gr}^{-1} \mathcal{H} \psi_{gr} = S^- + \omega S^3 + E_{gr}\,, \tag{8}$$

where the Lassalle operator (S^-) and Euler operator (S^3) are given by

$$S^- = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - (\nu - \delta) \sum_{j \neq k} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j}, \quad S^3 = \sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j}, \tag{9}$$

and

$$E_{gr} = \frac{N\omega}{2} \left[1 + (N-1)(\nu - \delta)\right]. \tag{10}$$

It is easy to see that the Lassalle operator and Euler operator, as defined in eqn.(9), satisfy the simple commutation relation: $[S^3, S^-] = -2S^-$. Using therefore the well known Baker-Hausdorff transformation we can remove the S^- part of the effective Hamiltonian $\mathcal{H}'$ and through some additional similarity transformations reduce it finally to the free oscillator model[7]

$$H_{free} = \mathcal{S}^{-1} \left(\mathcal{H}' - E_{gr}\right) \mathcal{S} = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{\omega^2}{2} \sum_{j=1}^{N} x_j^2 - \frac{\omega N}{2}, \tag{11}$$

where $\mathcal{S} = e^{\frac{1}{2\omega} S^-} e^{\frac{1}{4\omega} \nabla^2} e^{\frac{\omega}{2} \sum_{j=1}^{N} x_j^2}$ and $\nabla^2 = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$.

Due to similarity transformations in (8) and (11), one may naively think that the eigenfunctions of the extended Calogero model (5) can be obtained from those of free oscillators as: $\psi_{n_1, n_2, \cdots, n_N} = \psi_{gr} \mathcal{S} \left\{ \prod_{j=1}^{N} e^{-\frac{\omega}{2} x_j^2} H_{n_j}(x_j) \right\}$, where n_js are arbitrary non-negative integers and $H_{n_j}(x_j)$ denotes the Hermite polynomial of order n_j. However it is easy to check that, similar to the case of A_{N-1} Calogero model,[13] the action of $\mathcal{S}$ on free oscillator eigenfunctions leads to a singularity unless they are symmetrised with respect to all coordinates. Therefore, nonsingular eigenfunctions

of the extended Calogero model (5) can be obtained from the eigenfunctions of free oscillators as

$$\psi_{n_1,n_2,\cdots,n_N} = \psi_{gr}\ \mathcal{S}\Lambda_+ \left\{ \prod_{j=1}^{N} e^{-\frac{\omega}{2}x_j^2} H_{n_j}(x_j) \right\}, \tag{12}$$

where Λ_+ completely symmetrises all coordinates and thus projects the distinguishable many-particle wave functions to the bosonic part of the Hilbert space. Evidently, the eigenfunctions (12) will be mutually independent if the excitation numbers n_js obey the bosonic selection rule: $n_{j+1} \geq n_j$. Thus, in spite of the fact that the interacting Hamiltonian (5) is convertible to the free oscillator model, the need for symmetrization shows that the many-particle correlation is in fact inherent in this model. The eigenvalues of the Hamiltonian (5) corresponding to the states (12) will naturally be given by Ref. 7

$$E_{n_1,n_2,\cdots,n_N} = E_{gr} + \omega \sum_{j=1}^{N} n_j = \frac{N\omega}{2}\left[1 + (N-1)(\nu - \delta)\right] + \omega \sum_{j=1}^{N} n_j. \tag{13}$$

Since δ and ν are real parameters, the energy eigenvalues (13) are also real ones. Thus we interesting find that the nonhermitian PT invariant Hamiltonian (5) yields a real spectrum. Furthermore, it is evident that for all $n_j = 0$, the energy $E_{n_1,n_2,\cdots,n_N}$ attains its minimum value E_{gr}. At the same time, as can be easily worked out from eqn.(12), the corresponding eigenfunction reduces to ψ_{gr} (6). This proves that ψ_{gr} is indeed the ground state wave function for Hamiltonian (5) with eigenvalue E_{gr}.

It may be observed that the eigenfunctions (12) pick up a phase factor $(-1)^\nu$ under the exchange of any two particles. Therefore, ν represents the exchange statistics parameter for the extended Calogero model (5). By solving the quadratic eqn.(7), one can explicitly write down ν as a function of g and δ as

$$\nu = (\delta + \frac{1}{2}) \pm \sqrt{g + (\delta + \frac{1}{2})^2}. \tag{14}$$

For the purpose of obtaining real eigenvalues (13) as well as nonsingular eigenfunctions (12) at the limit $x_i \to x_j$, we have assumed at the beginning of this section that ν is a real positive parameter. This assumption leads to a restriction on the allowed values of the coupling constants g and δ in the following way. First of all, for the case $g < -(\delta + \frac{1}{2})^2$, eqn.(14) yields two imaginary solutions. Secondly, for the case $\delta < -\frac{1}{2}$, $0 > g > -(\delta + \frac{1}{2})^2$ eqn.(14) yields two real but negative solutions. Inequalities corresponding to these two cases represent two forbidden regions of (δ, g) plane which are excluded from our analysis.

For the case $g > 0$ with arbitrary value of δ, one gets a real positive and a real negative solution from eqn.(14). The real positive solution evidently leads to physically acceptable set of eigenfunctions and corresponding eigenvalues within this allowed region of (δ, g) plane. Finally we consider the parameter range $\delta >$

$-\frac{1}{2}$, $0 > g > -(\delta + \frac{1}{2})^2$, for which eqn.(14) yields two real positive solutions. It is easy to see that these two real positive solutions are related to each other through a 'reflection symmetry' given by $\nu \to 1 + 2\delta - \nu$. Consequently, for each point on the (δ, g) plane within this allowed parameter range, one obtains two different values of the exchange statistics parameter leading to two distinct sets of physically acceptable eigenfunctions and eigenvalues. Thus we curiously find that a kind of 'phase transition' occurs at the line $\delta = -\frac{1}{2}$ on the (δ, g) plane. For the case $\delta > -\frac{1}{2}$, exchange statistics parameter shows the reflection symmetry when g is chosen within an interval $-(\frac{1}{2} + \delta)^2 < g \leq 0$. On the other hand for the case $\delta \leq -\frac{1}{2}$, such reflection symmetry is lost for any possible value of g.

We have seen in this section that, similar to the case of A_{N-1} Calogero model, the extended model (5) can also be solved by mapping it to a system of free harmonic oscillators. So it is natural to enquire whether this extended model is directly related to the A_{N-1} Calogero model through some similarity transformation. Investigating along this line,[12] we find that

$$\Gamma^{-1}\mathcal{H}\Gamma = H' = \frac{1}{2}\sum p_j^2 + \frac{1}{2}\omega^2 \sum x_j^2 + g' \sum_{j \neq k}^{N} \frac{1}{(x_j - x_k)^2}\,, \tag{15}$$

where $\Gamma = \prod_{j<k}(x_j - x_k)^\delta$, and H' denotes the Hamiltonian of A_{N-1} Calogero model with 'renormalised' coupling constant given by $g' = g + \delta(1+\delta)$. Due to the existence of such similarity transformation, one may expect that the Hamiltonians $\mathcal{H}$ and H' always lead to exactly same eigenvalues. However it should be noted that, within a parameter range given by $\delta > 0$, $g > -\delta(1+\delta)$, there exists a positive solution of eqn.(7) satisfying the condition $\nu - \delta < 0$. Therefore, we can not get any lower bound for the corresponding energy eigenvalues (13) at $N \to \infty$ limit. On the other hand, the energy eigenvalues (2) of A_{N-1} Calogero model are certainly bounded from below for all possible choice of N and g. So there exists a parameter range within which the spectrum of extended Calogero model differs qualitatively from the spectrum of the original Calogero model. To explain this rather unexpected result, we first observe that the renormalised coupling constant g' would be a positive quantity within the above mentioned parameter range. Consequently, the corresponding exclusion statistics parameter ν', which is obtained by solving eqn.(3), has one positive and one negative solution. One usually throws away this negative solution of ν', since the corresponding eigenfunctions become singular at the limit $x_j \to x_k$. However, by using the relation (15), such singular eigenfunctions (denoted by $\psi'(x_1, x_2, \cdots, x_N)$) may now be used to construct the eigenfunctions of extended Calogero model (denoted by $\psi(x_1, x_2, \cdots x_N)$) as

$$\psi(x_1, x_2, \cdots, x_N) \;=\; \prod_{j<k}(x_j - x_k)^\delta \psi'(x_1, x_2, \cdots, x_N)\,. \tag{16}$$

It can be easily checked that, due to the existence of the factor $\prod_{j<k}(x_j - x_k)^\delta$, the r.h.s. of the above equation becomes nonsingular at the limit $x_j \to x_k$. Thus we curiously find that singular eigenfunctions of H' can be used to generate nonsingular

eigenfunctions of $\mathcal{H}$ through the relation (16). This shows that the similarity transformation (15) is a subtle one and, within a certain parameter range, eigenvalues of extended Calogero model will match with those of Calogero model (having renormalised coupling constant) only if the corresponding unphysical eigenfunctions are taken into account.

3. Connection with fractional statistics

We have mentioned in Sec.1 that GES can be realised microscopically in A_{N-1} Calogero model with hermitian Hamiltonian. The GES parameter for this Calogero model is a measure of 'level repulsion' of the quantum numbers generalising the Pauli exclusion principle. Now for exploring GES in the case of PT invariant model (5), we observe that eqn.(13) can be rewritten[7] exactly in the form of energy spectrum for N free oscillators as

$$E_{n_1,n_2\cdots n_N} = \frac{N\omega}{2} + \omega \sum_{j=1}^{N} \bar{n}_j \,, \tag{17}$$

where

$$\bar{n}_j = n_j + (\nu - \delta)(j-1) \tag{18}$$

are quasi-excitation numbers. However, from eqn.(18) it is evident that such quasi-excitation numbers are no longer integers and satisfy a modified selection rule: $\bar{n}_{j+1} - \bar{n}_j \geq \nu - \delta$. Since the minimum difference between two consecutive $\bar{n}_j$s is given by

$$\tilde{\nu} = \nu - \delta, \tag{19}$$

the spectrum of extended A_{N-1} Calogero model (5) satisfies GES with parameter $\tilde{\nu}$.[7] Several comments about this GES parameter are in order. It may be to noted that for $\delta \neq 0$, the GES parameter $\tilde{\nu}$ is different from the power index ν, which is responsible for the symmetry of the wave function. Therefore we may interestingly conclude that unlike Calogero model, the exclusion statistics for model (5) differs from its exchange statistics. Furthermore it is already noticed that, on a region of (δ, g) plane satisfying the inequalities $\delta > 0$, $g > -\delta(1+\delta)$, there exists a positive solution of eqn.(7) which yields a negative value of $\tilde{\nu}$. For this case, however, one does not get well defined thermodynamic relations at $N \to \infty$ limit and, therefore, can not interpret $\tilde{\nu}$ as the GES parameter.

By using eqn.(7) and (19), we find the relation

$$\tilde{\nu}^2 - \tilde{\nu} = g + \delta(\delta + 1), \tag{20}$$

which clearly describes a parabolic curve in the coupling constant plane (δ, g) for any fixed value of $\tilde{\nu}$. As a consequence of this, the competing effect of the independent coupling constants g and δ can make the GES feature of (5) much richer in comparison with the Calogero model. For example, while bosonic (fermionic) excitations in

the Calogero model occur only in the absence of long-range interaction, the quasi-excitations in (5) can behave as pure bosons (fermions) even in the presence of both the long-range interactions satisfying the constraint $\tilde{\nu}(\delta, g) = 0$ $(\tilde{\nu}(\delta, g) = 1)$. Both of these constraints lead to the same parabolic curve $g = -\delta(1 + \delta)$. A family of such parabolas with shifted apex points are generated for other values of $\tilde{\nu}$ and the lowest apex point is attained at $\tilde{\nu} = \frac{1}{2}$, where the quasi-excitations would behave as semions.

4. Conclusion

Here we construct a many-particle quantum system (5) by adding some nonhermitian but combined parity and time reversal (PT) invariant interaction to the A_{N-1} Calogero model. By using appropriate similarity transformations, we are able to map this extended Calogero model to a set of free harmonic oscillators and solve this model exactly. It turns out that this many-particle system with nonhermitian Hamiltonian yields a real spectrum. This fact supports the conjecture that the condition of hermiticity on a Hamiltonian can be replaced by the weaker condition of PT symmetry to ensure that the corresponding eigenvalues would be real ones. It is also found that the spectrum of extended Calogero model obeys a selection rule which leads to generalised exclusion statistics (GES).

However, this extended Calogero model exhibits some remarkable properties which are absent in the case of usual Calogero model. For example, we curiously find that the GES parameter for this extended Calogero model differs from the corresponding exchange statistics parameter. Moreover a 'reflection symmetry' of the exchange statistics parameter, which is known to exist for A_{N-1} Calogero model, can be found in the case of extended model only if the strength of PT invariant interaction exceeds a critical value.

Finally we note that, it is possible to obtain another exactly solvable many-particle quantum system by adding some nonhermitian but PT invariant interactions to the B_N Calogero model (associated with B_N Lie algebra).[12] Such a PT invariant model also leads to real spectrum with properties quite similar to the case of extended A_{N-1} Calogero model.

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THE ROTOR MODEL AND COMBINATORICS

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We examine the groundstate wavefunction of the rotor model for different boundary conditions. Three conjectures are made on the appearance of numbers enumerating alternating sign matrices. In addition to those occurring in the O($n = 1$) model we find the number $A_V(2m+1;3)$, which 3-enumerates vertically symmetric alternating sign matrices.

1. Introduction

The XXZ Heisenberg spin chain and the related six-vertex model stand as central pillars in the study of exactly solved models in statistical mechanics.[1,2] It has been known for many years that, with appropriate boundary conditions, their groundstate energy is trivial at the particular anisotropy value $\Delta = -1/2$. Only recently has it been realised that the corresponding groundstate wavefunction possesses some rather remarkable properties.[3–5] These observations extend to the related O(n) loop model[6,7] at $n = 1$.[4,8–10]

Consider first the periodic antiferromagnetic XXZ chain

$$H = -\frac{1}{2}\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z\right), \tag{1}$$

defined on an odd number N of sites. Here $(\sigma_j^x, \sigma_j^y, \sigma_j^z)$ are the Pauli spin matrices acting at site j. Normalize the smallest component of the groundstate wavefunction to be unity. Then at $\Delta = -1/2$ the largest component is conjectured to be given by[3]

$$A(m) = \prod_{j=0}^{m-1}\frac{(3j+1)!}{(m+j)!}, \tag{2}$$

for size $N = 2m+1$. The remarkable point being that $A(m)$ is the number of $m \times m$ alternating sign matrices.[11] The resulting sequence $A(m) = 1, 2, 7, 42, 429, 7436\ldots$ is also known to count other combinatorial objects.[12,13] Moreover, these numbers appear in the sum of all the groundstate wavefunction components. These observations remain to be proved.

An even number of sites and other boundary conditions have also been considered, both for the XXZ chain (twisted and closed[a] quantum symmetric bc's) and the O($n = 1$) loop model (periodic and closed bc's). These see the appearance of other well known numbers counting alternating sign matrices and related objects in different symmetry classes. For example, with the smallest component of the groundstate wavefunction again unity, the O($n = 1$) loop model with closed boundary conditions has largest component given by $A_{\mathrm{V}}(2m-1)$ for $N = 2m-1$ and $N_8(2m)$ for $N = 2m$. Here

$$A_{\mathrm{V}}(2m+1) = \prod_{j=0}^{m-1} (3j+2) \frac{(2j+1)!(6j+3)!}{(4j+2)!(4j+3)!} \tag{3}$$

is the number of $(2m+1) \times (2m+1)$ vertically symmetric alternating sign matrices and

$$N_8(2m) = \prod_{j=0}^{m-1} (3j+1) \frac{(2j)!(6j)!}{(4j)!(4j+1)!} \tag{4}$$

is the number of cyclically symmetric transpose complement plane partitions. The number $N_8(2m)$ is conjectured to be $A_{\mathrm{VH}}(4m+1)/A_{\mathrm{V}}(2m+1)$, where $A_{\mathrm{VH}}(4m+1)$ is the number of $(4m+1) \times (4m+1)$ vertically and horizontally symmetric alternating sign matrices.[14,15] Another quantity, which appears for periodic boundary conditions, is

$$A_{\mathrm{HT}}(2m) = A(m)^2 \prod_{j=0}^{m-1} \frac{3j+2}{3j+1}, \tag{5}$$

the number of $2m \times 2m$ half turn symmetric alternating sign matrices.

Further developments include the combinatorial interpretation of the elements of the O($n = 1$) loop model wavefunction in terms of link patterns[8–10] and the relation to a one-dimensional stochastic process.[10] There has been some progress attempting to prove these conjectures using Bethe Ansatz techniques.[16,17]

In this paper, we examine the groundstate wavefunction of the rotor model[18] discussed by Martins and Nienhuis. The rotor model is based on a variant of the Temperley-Lieb algebra, which underpins the six-vertex model, the O(n) model and the critical Q-state Potts model.[1,2,19] The rotor model is defined in Section 2, with our results presented in Section 3.

[a]The standard nomenclature for these bc's is open bc, but since these bc's are spin-conserving in the XXZ chain or loop reflecting in the O($n = 1$) model we find the term closed bc more appropriate, here reserving open bc for non-conserving boundary conditions.

Here we see the appearance of another number, $A_V(2m+1;3)$, which is the 3-enumeration of $(2m+1) \times (2m+1)$ vertically symmetric alternating sign matrices, or equivalently, the number of vertically symmetric 6-vertex configurations with domain wall boundary conditions and $\Delta = -1/2$. It is given by

$$A_V(2m+1;3) = \frac{3^{m(m-3)/2}}{2^m} \prod_{j=1}^{m} \frac{(j-1)!(3j)!}{j(2j-1)!^2} = 1, 5, 126, 16038, \ldots. \tag{6}$$

In general, the x-enumeration of alternating sign matrices in the terminology of Kuperberg,[15] is equivalent to the enumeration of six-vertex configurations with domain wall boundaries with $\Delta = 1 - x/2$ and at the symmetric point with respect to the spectral parameter.

We give some concluding remarks in Section 4.

2. The rotor model

We suggest that the remarkable observations of this $O(n = 1)$ model are related to the combination of two key properties, namely solvability and the absence of finite size corrections to the groundstate energy. Now the $O(n = 1)$ model is not unique in this combination. Recently Martins and Nienhuis [18] introduced a model that shares the same two properties. In this so-called rotor model a set of loops covers all the edges of the square lattice precisely twice. At the vertices all the loops make a turn of $\pi/2$ which permits four types of vertices as displayed in Figure 1.

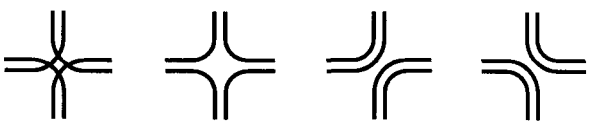

Fig. 1. Vertices of the rotor model.

A natural interpretation is that the loops are trajectories of particles, and that the two loop segments visiting the same edge are traversed in opposite directions. Thus the four kinds of vertices shown in Figure 1 behave as scatterers: right (R) and left (L) rotors, at which the particles always turn right and left respectively, and ascending (A) and descending (D) diagonal mirrors at which the particles get reflected. To clearly display the scatterers we propose that the particles always follow the left hand side of the road, as is customary in Australia where this paper was conceived.

In a different interpretation the two loop segments at the same edge are the trajectories of different kinds of particles, traversed in either direction. Then the scatterers can all be interpreted as double mirrors on each site, each reflecting one kind of particle and transmitting the other. At the R and L sites these mirrors

are placed crosswise, AD and DA respectively, while at the original ascending and descending mirrors, the double mirrors are placed parallel, AA and DD respectively. This alternate interpretation will not affect the distributions of trajectories in an infinite system, but it will result in changes on some finite systems.

Martins and Nienhuis solved this model by means of the Yang-Baxter equation when these scatterers occur with the respective weights

$$\begin{aligned} \omega_{\mathrm{R}} &= \omega_{\mathrm{L}} = \sin u \cos(2\pi/3 - u), \\ \omega_{\mathrm{A}} &= \sin(\pi/3 - u) \cos(2\pi/3 - u), \\ \omega_{\mathrm{D}} &= -\sin u \cos(\pi/3 - u). \end{aligned} \tag{7}$$

independently at each vertex. In this paper we consider this model with periodic boundary conditions (pbc) and with closed boundaries at which the trajectories are reflected. We will be interested in the structure of the groundstate eigenvector. Since the transfer matrix as a function of u forms a commuting family, the groundstate is independent of u. Then it is convenient to consider the Hamiltonian, found (up to a constant) as the logarithmic derivative of the transfer matrix with respect to u at $u = 0$:

$$H = \sum_i 3 - R_i - L_i - E_i. \tag{8}$$

For system size N the operators R, L and E are shown in terms of the loops in Figure 2.

Fig. 2. Generators.

Martins and Nienhuis showed that the operators L_{2i} and R_{2i-1} generate a Temperley-Lieb (TL) algebra, and so do the operators L_{2i-1} and R_{2i}. In periodic systems of even size, and in bounded systems these two TL algebras commute with each other. What changes the physics is the presence in the Hamiltonian of the term $E_i = R_i L_i$. Also the E_i by themselves generate a TL algebra. In odd, periodic systems the odd and even sites cannot be distinguished. In this case the L and the R together form a TL algebra of $2N$ sites.

When the system is odd and periodic, the interpretation of the R and L vertices as rotors or alternatively as crossing mirrors, will naturally result in different pbc. The rotor interpretation permits closed trajectories that wind the cylinder twice. In the alternative interpretation no closed winding trajectories are possible, and the odd system must have two unmatched terminals. In this paper we follow the latter interpretation.

The states of the model are the pairings of those terminals that are connected by a trajectory in the 'past' half of the strip or cylinder. When the system is periodic, one may distinguish the side of the cylinder along which the trajectory runs: a connection between site 1 and site N may pass all sites $2, \ldots N-1$, or it may simply connect site N to site $N+1$ which is identified to 1. These two states can be distinguished, in which case we speak of pbc *per se*, or they may be identified, for which we reserve the phrase pbc with identified connectivities.

3. Results for the groundstate wavefunction

The groundstate wavefunction of the Hamiltonian (8) satisfies the eigenvalue equation $H\psi_0 = 0$. In this section we formulate three conjectures regarding ψ_0 for the different types of boundary conditions discussed in Section 2.

Conjecture 1: *For closed boundary conditions, if the smallest element of the rotor model groundstate wavefunction for $N = 2m-1$ is normalized to $A_{\mathrm{V}}(2m-1;3)$, then all of its elements are integers and the sum of its elements is given by $S(2m-1) = 3^{(m-1)^2} N_8(2m)$. For $N = 2m$, normalize the groundstate wavefunction to the smallest integer such that all elements are integers, the sum of the elements is given by $S(2m) = 3^{2\theta_m} A_{\mathrm{V}}(2m+1)$, where $\theta_m = 0,1,3,6,9 = \lfloor (m-1)(m+2)/3 \rfloor$ for $m = 1, \ldots, 5$.*

This conjecture is based on the results presented in Table 1 and was checked up to $N = 10$.

Conjecture 2: *For periodic boundary conditions, normalize the smallest element of the rotor model groundstate wavefunction to the smallest integer such that all elements are integer. The sum of its elements is then given by $S(2m-1) = 3^{3m} A_{\mathrm{V}}(2m+1;3)^2$ for odd system sizes and by $S(2m) = 3^{m^2} A_{\mathrm{HT}}(2m)$ for even system sizes.*

This conjecture is based on the results presented in Table 2 and was checked up to $N = 9$.

Conjecture 3: *For periodic boundary conditions and identified connectivities, normalize the smallest element of the rotor model groundstate wavefunction to the smallest integer such that all elements are integer. The sum of its elements is then given by $S(2m) = 3^{\theta_m} A(m)$, where $\theta_m = 0,1,3,6,9,13 = \lfloor (m-1)(m+2)/3 \rfloor$ for $m = 1, \ldots, 6$.*

This conjecture is based on the results presented in Table 3 and was checked up to $N = 12$.

4. Discussion

In this paper we have examined the groundstate wavefunction of the rotor model for three different boundary conditions. As for the $\mathrm{O}(n=1)$ model, numbers known to

Table 1. Groundstate wavefunctions of the rotor model with closed boundaries. Note that by $\psi_0 = (2,1)$ with multiplicity $(2,2)$ we mean $\psi_0 = (2,2,1,1)$.

N	m	ψ_0	multiplicity	$S_N^{(1)}$
1	1	(1)	(1)	1
2	1	(1)	(1)	1
3	2	(2,1)	(2,2)	6
4	2	(14,5,4)	(1,1,2)	27
5	3	(113, 111, 55, 31, 25, 21, 19, 11, 5)	(2, 1, 4, 2, 4, 2, 4, 4, 2)	891
6	3	(4760, 1440, 1192, 1028, 601, 565, 326, 310, 126, 121, 86)	(1, 2, 4, 1, 4, 2, 2, 2, 1, 2, 4)	18954

Table 2. Groundstate wavefunctions of the rotor model with periodic boundaries.

N	m	ψ_0	multiplicity	$S_N^{(1)}$
1	1	(1)	(1)	1
2	1	(2,1)	(2,2)	6
3	2	(5,2)	(3,6)	27
4	2	(118, 35, 25, 22, 20, 5, 4)	(2, 2, 8, 4, 8, 8, 4)	810
5	3	(1036, 463, 208, 143, 127, 122, 65, 22, 10)	(5, 10, 10, 20, 5, 10, 20, 10, 10)	18225

Table 3. Groundstate wavefunctions of the rotor model with periodic boundaries and identified connectivities.

N	m	ψ_0	multiplicity	$S_N^{(1)}$
2	1	(1)	(1)	1
4	2	(2,1)	(2,2)	6
6	3	(26, 9, 7, 2)	(2, 3, 14, 6)	189
8	4	(1798, 486, 410, 267, 234, 232, 165, 106, 90, 81, 76, 70, 56, 45, 20, 9, 4)	(2, 8, 16, 2, 16, 16, 8, 16, 4, 16, 8, 8, 16, 32, 16, 8, 4)	30618

enumerate equally weighted alternating sign matrices appear in the normalization of the wavefunction. For the rotor model we also see the number $A_V(2m+1;3)$, enumerating alternating sign matrices in which the minus signs have weight 3.[15]

We find it quite surprising that the conjectures in Section 3 can be formulated at all. They are a result of the normalizations factoring into relatively small primes and thus enabling their recognition. This property appears to be absent for other boundary conditions, for example, pbc in the rotor interpretation for odd system sizes. It is even more remarkable that these numbers have a well known combinatorial meaning.

Acknowledgements

It is a great pleasure to dedicate this paper to Fred Wu on the occasion of his 70th birthday. This work has been supported by the Australian Research Council and the Dutch foundation 'Fundamenteel Onderzoek der Materie' (FOM).

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PHASE TRANSITIONS IN THE TWO-DIMENSIONAL O(3) MODEL

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We use Monte Carlo, transfer-matrix and finite-size scaling methods to investigate two-dimensional O(n) models with $n > 2$, in particular the case $n = 3$ which includes the classical Heisenberg model. Depending on the type of interaction and the lattice structure, two different types of phase transitions are present. One type resembles the hard-hexagon transition and occurs in the loop representation of the honeycomb O(n) model. The other type is a first-order transition which occurs for spin-spin interactions that are strongly nonlinear in the neighbor-spin products. When the nonlinearity is decreased, the first-order line ends in a critical point. The existence of the first-order line is in agreement with mean-field theory as well as with high- and low-temperature approximations.

1. Introduction

The O(n) spin model represents a system of interacting n-dimensional vectors $\vec{s} \equiv (s_1, s_2, \cdots, s_n)$ on a lattice. The O(n) symmetry implies that the Hamiltonian is invariant under rotations in the space of the spin vectors. We consider the case of O(n) symmetric pair interactions

$$\mathcal{H}/k_{\mathrm{B}}T = -\sum_{<i,j>} h(\vec{s}_i.\vec{s}_j) \tag{1}$$

where the sum is on all pairs of nearest neighbors, and h is an arbitrary function with implicit temperature dependence. The 'linear' case, i.e. $h(x) = Kx$ where K is the coupling constant, includes the classical XY model ($n = 2$) and the classical Heisenberg model ($n = 3$). For reasonable choices of the function h, in particular monotonically increasing functions, it is plausible that the model belongs to the same universality class as the linear model.

The question whether or not two-dimensional O(n) models with $n > 2$, and in particular with $n = 3$, undergo a phase transition at a sufficiently low temperature has received considerable attention;[1-6] see also references therein. These papers contain conflicting answers. However, according to the prevailing interpretation, ordering transitions are absent in the O(n) model with $n > 2$. The main argument relies on the spin-wave result[1,3] by Bloch that a spontaneously magnetized state cannot exist at a nonzero temperature. This does not qualify as a proof for the absence of a transition: the spin-wave argument applies as well to the XY ($n = 2$) model where a phase transition[7] is known to occur (but not to a spontaneously magnetized state). The latter transition is however linked to topological excitations (vortices) which lack relevance for $n = 3$. This, together with an exact result of Kunz and Wu[4] which excludes phase transitions in a part of the $n > 2$ parameter space, forms the basis of the above-mentioned prevailing interpretation.

However, here we describe two sorts of transition for $n > 2$. The first type, which occurs in the O(n) loop model on the honeycomb lattice, is described in section 2. It is unphysical in the spin representation, because negative Boltzmann weights occur. It is thus consistent with the hypothesis that phase transitions are absent in two-dimensional *spin* models with $n > 2$. However, in section 3 we describe a phase transition in a genuine Heisenberg-type O(3) model. The phase transition does not lead to a long-range ordered state (in the sense of a nonzero spontaneous magnetization) and is therefore consistent with the spin-wave theory.

2. Phase Transition in the Loop Model

The 'loop' version of the O(n) model is defined by the choice $h(x) = \log(1 + ax)$, where the parameter a is an inverse-temperature-like parameter, and the normalization $\vec{s}_i.\vec{s}_i = n$. The model on the honeycomb lattice can be mapped[5] onto a gas of nonintersecting loops running over the edges of the honeycomb lattice. Each edge covered by a loop carries a Boltzmann weight a, and each loop a weight n. The loop representation has enabled exact solutions along special lines in the n, a plane for $n \leq 2$ [8-11] and for $n \geq 2$.[12] From these solutions we know that for $n \leq 2$ an ordering transition occurs at finite values of x, and that for $n > 2$ the loop model is in a long-range ordered state for $x = \infty$. This state is not of the 'ferromagnetic' type, but chooses between 3 sublattices, and reminds of the hard-hexagon model.

In the absence of exact solutions in most of the n, x plane, we have applied Monte Carlo and transfer-matrix techniques.[13] Surprisingly, a transition to the long-range ordered state was found at *finite* values $x < \infty$. The hard-hexagon-like critical line spans the range $2 < n < \infty$. The resulting phase diagram is shown in fig. 1. Also shown are the boundary of the 'physical region' of the spin model, where the energy $h(\vec{s}_i.\vec{s}_j)$ is real for all $\vec{s}_i.\vec{s}_j$ (curved line in the middle), and a region where a transition is rigorously excluded (above the curved line on the far right). The latter line is based on the work by Kunz and Wu;[4] see also an erratum.[13]

We observe that the newly found critical line indeed avoids the excluded region.

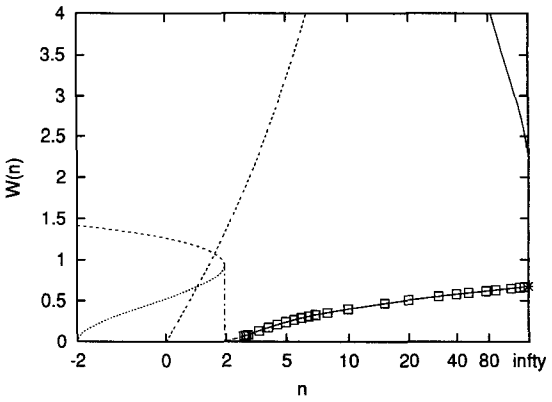

Fig. 1. Phase diagram of the O(n) model on the honeycomb lattice in the n-a plane. The horizontal scale is chosen as $1 - 8/(n + 10)$ in order to show the whole range up to $n = \infty$. The vertical scale displays $W(n) = 1/[(n + 10)^{1/6}a]$. The data points show our results for the newly found phase transition. The curve in the range $-2 < n \leq 2$ shows an exact solution. The region on the right hand side of curve ending at $n = W = 0$ is unphysical in the spin representation. Rigorous arguments exclude phase transitions in the region indicated at the upper right.

It is completely embedded in the unphysical region of the spin model, but it is physical in the language of the O(n) loop model.

3. The Strongly Nonlinear O(3) Model

We use Eq. (1) for the Heisenberg case $n = 3$ with a spin-spin interaction

$$h(\vec{s}_i \cdot \vec{s}_j) = 2K[(1 + \vec{s}_i \cdot \vec{s}_j)/2]^p \tag{2}$$

with spins normalized to length 1. The parameter p determines the degree of nonlinearity of the energy function h. This form is chosen as to avoid powers of negative numbers, and to limit the energy range to $2K$, as in the linear case.

3.1. *Mean-field theory*

Consider a spin $\vec{s}_i$, interacting with z neighbors. Denote the average magnetization of the spin system as m, say along the x-axis. The local energy is

$$E_{\text{loc}} = -2zK[(1 + \vec{s}_i \cdot \vec{m})/2]^p = -2zK[(1 + xm)/2]^p \tag{3}$$

and the thermal average of the x-component of $\vec{s}_i$ satisfies

$$\langle x \rangle = \int_{-1}^{1} x e^{-E_{\text{loc}}} dx / \int_{-1}^{1} e^{-E_{\text{loc}}} dx \tag{4}$$

For large enough K, the self-consistency equation $\langle x \rangle = m$ has solutions at nonzero m. While m is a decreasing function of K, this function depends qualitatively on

p. For small enough values of p, m decreases continuously to 0 at the critical point $K = K_c$. We thus find K_c by solving $\partial\langle x\rangle/\partial m = 1$ which leads to

$$K_c = 2^{p-3}3/p \tag{5}$$

For larger values p, $\langle x\rangle$ increases faster than linear with m for small m, but still levels off at large m. The nozero solutions of $\langle x\rangle = m$ thus become duplicate, i.e., the transition turns first order for large p. These two distinct ranges of p are separated by the tricritical point, which can be determined by solving for K and p in $\partial\langle x\rangle/\partial m = 1$ and $\partial^3\langle x\rangle/\partial m^3 = 0$ at $m = 0$. The second equation expresses the absence of the lowest order of nonlinearity of $\langle x\rangle$ as a function of m. The tricritical coordinates are $p_{\rm tri} = (-1 + \sqrt{33})/2$ and $K_{\rm tri} = 2^{p_{\rm tri}-3}3/p_{\rm tri}$.

For $p > p_{\rm tri}$, the first-order transition point follows by equating the areas enclosed by the $\langle x\rangle$ vs. m curve and the $\langle x\rangle = m$ line. Thus we have determined the first order line numerically; the resulting phase diagram is shown in fig. 2.

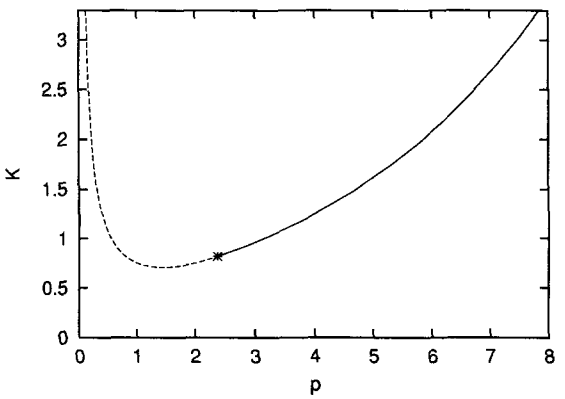

Fig. 2. Mean-field phase diagram of the nonlinear O(3) model on the square lattice, in the K versus p plane. The line of phase transitions consists of two parts: a continuous transition at small p, and a first-order part. The two parts are separated by a tricritical point (asterisk).

3.2. *High- and low-temperature approximations*

Estimates of the location of a first-order transition(if any) can also be obtained from intersections of high- and low-temperature approximations of the free energy. Neglecting loop diagrams in the high-temperature expansion the lattice effectively reduces to the Bethe lattice, for which we obtain the free energy via a transfer-matrix-like approach. The partition function $Z_{\rm b}$ 'per bond' between spins $\vec{s}$ and $\vec{t}$ equals

$$Z_{\rm b} = \int {\rm d}\vec{s}\, \exp\{2K[(1 + \vec{s}.\vec{t})/2]^p\} \tag{6}$$

which is independent of $\vec{t}$. Expansion of the exponential function yields

$$Z_{\mathrm{b}} = 4\pi \sum_{k=0}^{\infty} \frac{(2k)^k}{(1+pk)k!} \tag{7}$$

where the prefactor accounts for the spin degrees of freedom and the sum for the spin-spin interaction. The partition function of a Bethe lattice with N spins, each interacting with z neighbors, and $zN/2$ bonds follows as

$$Z_{\mathrm{BL}} = (4\pi)^N \left(\sum_{k=0}^{\infty} \frac{(2k)^k}{(1+pk)k!} \right)^{zN/2} \tag{8}$$

For $z = 4$, the high-temperature approximation of the free energy is thus

$$\frac{F_{\mathrm{HT}}}{NkT} = -\log(4\pi) - 2\log\left(\sum_{k=0}^{\infty} \frac{(2k)^k}{(1+pk)k!} \right) \tag{9}$$

We use a low temperature approximation for spins almost aligned along the z axis: $\vec{s}_i = \left(s_i^x, s_i^y, \sqrt{1 - s_i^{x2} - s_i^{y2}} \right)$ where the x and y components are small. Small deviations between neighbors i and j increase the energy per bond

$$E_{ij}/kT + 2K = \frac{1}{2} pK[(s_i^x - s_j^x)^2 + (s_i^y - s_j^y)^2] \tag{10}$$

i.e. the Gaussian model applies to this quadratic form. After a Fourier transformation it is straightforward to obtain the partition function; the free energy follows as

$$\frac{F_{\mathrm{LT}}}{NkT} = -4K - \frac{\log Z_{\mathrm{G}}^2}{N} =$$

$$-4K - \log(4\pi) + \log(8pK) + \frac{1}{N} \sum_{\vec{k}} \log[(\sin \frac{1}{2} k_x)^2 + (\sin \frac{1}{2} k_y)^2] \tag{11}$$

For large N the sum satisfies $\frac{1}{N} \sum_{\vec{k}} \log[(\sin \frac{1}{2} k_x)^2 + (\sin \frac{1}{2} k_y)^2] \simeq -0.2200507 \cdots$.

The low- and high-temperature approximations of the free energy are found to intersect, and thus predict the approximate location of a possible first-order transition line. These intersections were found numerically, and shown in fig. 3.

3.3. *Monte-Carlo results*

The model defined by Eqs. (1) and (2) was investigated by a conventional Monte Carlo algorithm with local spin updates. Randomly chosen orientations for the spin vectors were accepted or rejected with Metropolis-type probabilities. The autocorrelation times are found to increase considerably at low temperatures, especially for system sizes L exceeding about 100. The efficiency of the algorithm decreases even further at high values of p where the acceptance ratio becomes small.

Nevertheless we could resolve the phase diagram. No signs of a phase transition were found for $p = 1$, i.e. the linear Heisenberg model. But for larger p, pronounced

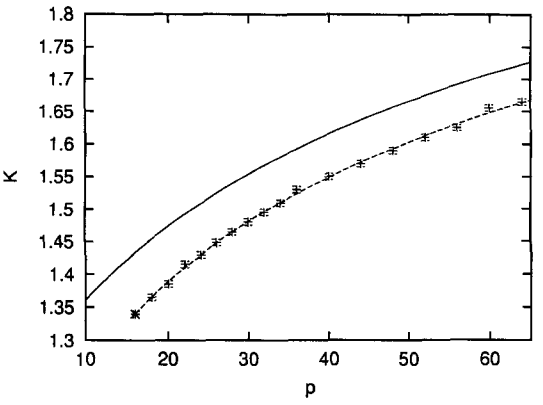

Fig. 3. Monte Carlo results for the phase diagram of the two-dimensional O(n) spin model on the square lattice, in the K versus p plane. Results from the intersections of low- and high-temperature approximations of the free energy are included as well (full curve).

maxima in the heat capacity appear, and for $p \geq 16$, we find numerical evidence for a divergence of the heat capacity. For $p > 18$, the simulations reveal a jump in the energy as a function of K, and a clear hysteresis effect. The first-order character becomes even stronger at larger p. The transition for $p > 20$ was found by Monte Carlo runs starting from a spin configuration of which one half was fully aligned, and the other half filled with randomly chosen spins. For $p < 20$ we determined the location $K_{\max}(p, L)$ of the heat capacity maximum $c_{\max}(p, L)$ as a function of K for system sizes up to $L = 48$, and extrapolated to $L = \infty$. The heat capacity does not seem to diverge for $p < 16$. For $p = 16$ we observe a divergence approximately as $L^{7/4}$, which indicates the presence of an Isinglike critical point. For $p > 16$ the divergence agrees with first-order behavior $c_{\max}(p, L) \propto L^2$. The Monte Carlo data are included in fig. 3.

4. Discussion

We have provided evidence for two types of phase transitions in O(n) models with $n > 2$. The first type is unphysical in the spin language, and depends essentially on the underlying lattice structure. The second type is, however, found in a pure spin system, in a conspicuous disagreement with expectations formulated in the literature. We review the evidence presented above.

In low-dimensional models, mean-field theory tends to predict continuous phase transitions where they do not exist. The example found in section 3.1 should thus not be taken too seriously. The predicted first-order transition is more credible, because the strength of the predicted discontinuity increases at large p, and the role of fluctuations may thus be reduced. Another defect of mean-field theory is that it is based on an order parameter m that is actually zero.[1,3] However, the

long-wavelength spin waves, which are responsible for the suppression of m, hardly affect correlations at distances of a few lattice units. The mean-field result can thus still be regarded as suggestive of a first-order transition at large p.

Likewise, the result of the high- and low-temperature approximations is less than compelling. Nevertheless, here also the predicted energy jump increases with p: the two free-energy branches are pushed far away, mimicking high- and low-temperature configurations with only limited fluctuations. This lends some support to our approximation. The resulting first-order line is in a reasonable qualitative agreement with Monte Carlo results (fig. 3). It tends to become better at large p. The Monte Carlo runs provided a clear first-order picture; the numerical errors are very small in comparision to the differences with the two analytic approximations for the first-order line. The Monte Carlo results are clearly superior. Finally we mention that similar transitions may occur for larger values of n, and that Monte Carlo results of Domany et al.[14] for the analogous strongly nonlinear O(2) model showed that the Kosterlitz-Thouless transition is preempted by a first order one.

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THE 8V CSOS MODEL AND THE sl_2 LOOP ALGEBRA SYMMETRY OF THE SIX-VERTEX MODEL AT ROOTS OF UNITY

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We review an algebraic method for constructing degenerate eigenvectors of the transfer matrix of the eight-vertex Cyclic Solid-on-Solid lattice model (8V CSOS model), where the degeneracy increases exponentially with respect to the system size. We consider the elliptic quantum group $E_{\tau,\eta}(sl_2)$ at the discrete coupling constants: $2N\eta = m_1 + im_2\tau$, where N, m_1 and m_2 are integers. Then we show that degenerate eigenvectors of the transfer matrix of the six-vertex model at roots of unity in the sector $S^Z \equiv 0 \pmod N$ are derived from those of the 8V CSOS model, through the trigonometric limit. They are associated with the complete N strings. From the result we see that the dimension of a given degenerate eigenspace in the sector $S^Z \equiv 0 \pmod N$ of the six-vertex model at Nth roots of unity is given by $2^{2S^Z_{max}/N}$, where S^Z_{max} is the maximal value of the total spin operator S^Z in the degenerate eigenspace.

1. Introduction

Recently, it has been explicitly discussed that the transfer matrix of the six-vertex model at roots of unity has the symmetry of the sl_2 loop algebra.[1–5] Let us consider the XXZ spin chain under the periodic boundary conditions

$$H_{XXZ} = -J\sum_{j=1}^{L}\left(\sigma_j^X\sigma_{j+1}^X + \sigma_j^Y\sigma_{j+1}^Y + \Delta\sigma_j^Z\sigma_{j+1}^Z\right). \tag{1}$$

Here the parameter Δ is related to the q variable of the quantum group $U_q(sl_2)$ as

$$\Delta = \frac{1}{2}(q + q^{-1}). \tag{2}$$

When $q^{2N} = 1$, it was shown[1] that the XXZ Hamiltonian commutes with the generators of the sl_2 loop algebra, which is an infinite dimensional algebra. Furthermore, it was shown[1] by the Jordan-Wigner method for $N = 2$ and numerically for general N that the dimensions of the degenerate eigenvectors are given by some powers of 2, which increase exponentially with respect to the system size L.

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The exponential degeneracy of the sl_2 loop algebra should be important for the problem of the "completeness of the Bethe ansatz eigenvectors". In fact, the sl_2 loop algebra symmetry has not been considered in the standard arguments of the string hypothesis.[6,7] Thus, it seems that it is still open whether we can construct 2^L linearly independent eigenvectors of the XXZ spin chain at roots of unity for general L. The question should be related to so called singular Bethe ansatz solutions.[8] In fact, it is numerically confirmed that the standard solutions of the Bethe ansatz equations determine only eigenvectors which have the highest weights of the sl_2 loop algebra.[2] Furthermore, some important properties of complete N strings have been discussed in association with the sl_2 loop algebra.[2-4]

Interestingly, it was numerically suggested that the transfer matrix of the eight-vertex model at the discrete coupling parameters should have the degenerate eigenvectors corresponding to the degeneracy of the sl_2 loop algebra.[1] Furthermore, it has been recently shown that some degenerate eigenspace of the eight-vertex model has dimension of $N2^{L/N}$ if L/N is an even integer.[5] Let us consider the XYZ Hamiltonian under the periodic boundary conditions[9,10]

$$H_{XYZ} = -\sum_{j=1}^{L} \left(J_X \sigma_j^X \sigma_{j+1}^X + J_Y \sigma_j^Y \sigma_{j+1}^Y + J_Z \sigma_j^Z \sigma_{j+1}^Z \right) , \tag{3}$$

where the coupling constants J_X, J_Y and J_Z are given by

$$J_X = J(1 + k\,\mathrm{sn}^2(2\eta)) , \quad J_Y = J(1 - k\,\mathrm{sn}^2(2\eta)) , \quad J_Z = J\,\mathrm{cn}(2\eta)\mathrm{dn}(2\eta) . \tag{4}$$

Here $\mathrm{sn}(z)$, $\mathrm{cn}(z)$ and $\mathrm{dn}(z)$ denote the Jacobian elliptic functions with elliptic modulus k. We have called 2η the coupling parameter of the model. The number N has been related to 2η by $2N\eta = 2m_1 K + im_2 K'$. The symbols K and K' denote the complete elliptic integrals of the first and second kinds, respectively.

In this paper, we discuss an algebraic construction of degenerate eigenvectors of the eight-vertex cyclic Solid-on-Solid model[11-13] (8V CSOS model), which is a variant of the eight-vertex Restricted Solid-on-Solid model (ABF model) Then, we show that through some limit, they give the degenerate eigenvectors of the six-vertex model in the sector $S^Z \equiv 0 \pmod N$ consisting of the complete N strings.

2. The sl_2 loop algebra symmetry of the XXZ spin chain

Let us consider representations of the generators of $U_q(sl_2)$ on the Lth tensor product of spin 1/2 representations.

$$q^{S^Z} = q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2} \tag{5}$$

$$S^{\pm} = \sum_{j=1}^{L} S_j^{\pm} = \sum_{j=1}^{L} q^{\sigma^Z/2} \otimes \cdots q^{\sigma^Z/2} \otimes \sigma_j^{\pm} \otimes q^{-\sigma^Z/2} \otimes \cdots \otimes q^{-\sigma^Z/2} \tag{6}$$

Let us introduce some symbols: $[n] = (q^n - q^{-n})/(q - q^{-1})$ for $n > 0$ and $[0] = 1$; $[n]! = \prod_{k=1}^{n} [k]$. Setting

$$S^{\pm(N)} = \lim_{q^{2N} \to 1} (S^{\pm})^N / [N]! \tag{7}$$

the operators $S^{\pm(N)}$ are non-vanishing and we have

$$S^{\pm(N)} = \sum_{1\le j_1<\cdots<j_N\le L} q^{\frac{N}{2}\sigma^Z}\otimes\cdots\otimes q^{\frac{N}{2}\sigma^Z}\otimes\sigma^{\pm}_{j_1}\otimes q^{\frac{(N-2)}{2}\sigma^Z}\otimes\cdots\otimes q^{\frac{(N-2)}{2}\sigma^Z}$$
$$\otimes\sigma^{\pm}_{j_2}\otimes q^{\frac{(N-4)}{2}\sigma^Z}\otimes\cdots\otimes\sigma^{\pm}_{j_N}\otimes q^{-\frac{N}{2}\sigma^Z}\otimes\cdots\otimes q^{-\frac{N}{2}\sigma^Z} \quad . \tag{8}$$

The study of the symmetries of the XXZ Hamiltonian under periodic boundary conditions at roots of unity was initiated in Ref.:[15] $S^{\pm(N)}$ commute with the Hamiltonian (1) when S^Z/N is an integer and $q^{2N}=1$ holds. However, there exists a much larger symmetry algebra than that of $S^{\pm(N)}$.[1] We remark that the XXZ Hamiltonian is associated with the affine quantum group $U_q(\hat{sl}_2)$. For instance, we may consider the following:

$$T^{\pm} = \sum_{j=1}^{L} T^{\pm}_j = \sum_{j=1}^{L} q^{-\sigma^z/2}\otimes\cdots q^{-\sigma^z/2}\otimes\sigma^{\pm}_j\otimes q^{\sigma^z/2}\otimes\cdots\otimes q^{\sigma^z/2}\,, \tag{9}$$

which is also obtained from $S^{\pm}$ by the replacement $q\to q^{-1}$. When $q^{2N}=1$, we define $T^{\pm(N)}$ similarly as in eq. (7).

Let $T_{6V}(v)$ denotes the (inhomogeneous) transfer matrix of the six-vertex model. Then we can show the (anti) commutation relations when $S^Z\equiv 0 \pmod N$[1]

$$S^{\pm(N)}T_{6V}(v) = q^N T_{6V}(v)S^{\pm(N)}, \qquad T^{\pm(N)}T_{6V}(v) = q^N T_{6V}(v)T^{\pm(N)} \tag{10}$$

and therefore in the sector $S^Z\equiv 0 \pmod N$ we have

$$[S^{\pm(N)}, H] = [T^{\pm(N)}, H] = 0. \tag{11}$$

Let us discuss the symmetry algebra. With the following identification[1]

$$e_0 = S^{+(N)}, \quad f_0 = S^{-(N)}, \quad e_1 = T^{-(N)},$$
$$f_1 = T^{+(N)}, \quad t_0 = -t_1 = -(-q)^N S^z/N\,, \tag{12}$$

we can show that they satisfy the defining relations of the sl_2 loop algebra:

$$[S^{+(N)}, T^{+(N)}] = [S^{-(N)}, T^{-(N)}] = 0\,, \tag{13}$$

$$[S^{\pm(N)}, S^Z] = \pm N S^{\pm(N)}, \qquad [T^{\pm(N)}, S^Z] = \pm N T^{\pm(N)}\,, \tag{14}$$

$$S^{+(N)3}T^{-(N)} - 3S^{+(N)2}T^{-(N)}S^{+(N)} + 3S^{+(N)}T^{-(N)}S^{+(N)2} - T^{-(N)}S^{+(N)3} = 0\,,$$
$$S^{-(N)3}T^{+(N)} - 3S^{-(N)2}T^{+(N)}S^{-(N)} + 3S^{-(N)}T^{+(N)}S^{-(N)2} - T^{+(N)}S^{-(N)3} = 0\,,$$
$$T^{+(N)3}S^{-(N)} - 3T^{+(N)2}S^{-(N)}T^{+(N)} + 3T^{+(N)}S^{-(N)}T^{+(N)2} - S^{-(N)}T^{+(N)3} = 0\,,$$
$$T^{-(N)3}S^{+(N)} - 3T^{-(N)2}S^{+(N)}T^{-(N)} + 3T^{-(N)}S^{+(N)}T^{-(N)2} - S^{+(N)}T^{-(N)3} = 0\,, \tag{15}$$

and in the sector $S^z\equiv 0 (\mathrm{mod}\ N)$ we have

$$[S^{+(N)}, S^{-(N)}] = [T^{+(N)}, T^{-(N)}] = -(-q)^N\frac{2}{N}S^z. \tag{16}$$

The loop algebras with higher ranks are also discussed for some vertex models.[16]

3. The algebraic Bethe ansatz of the elliptic quantum group $E_{\tau,\eta}(sl_2)$

The elliptic algebra $E_{\tau,\eta}(sl_2)$ is an algebra generated by meromorphic functions of a variable h and the matrix elements of a matrix $L(z,\lambda)$ with non-commutative entries,[17,18] which satisfy the Yang-Baxter relation with a dynamical shift

$$\begin{aligned} &R^{(12)}(z_{12},\lambda-2\eta h^{(3)})L^{(1)}(z_1,\lambda)L^{(2)}(z_2,\lambda-2\eta h^{(1)}) \\ &=L^{(2)}(z_2,\lambda)L^{(2)}(z_1,\lambda-2\eta h^{(2)})R^{(12)}(z_{12},\lambda)\quad . \end{aligned} \tag{17}$$

Here h is a generator of the Cartan subalgebra $\mathbf{h}$ of sl_2. Drinfeld's quasi-Hopf algebra gives a natural framework for the dynamical Yang-Baxter relation, which can be derived from the standard quantum group $U_q(\hat{sl}_2)$ through the twist.[19,20]

The R-matrix of (17) is essentially that of the ABF model[14] (the 8V RSOS model). Let V be the two-dimensional complex vector space with the basis $e[1]$ and $e[-1]$. Here we denote $e[-1]$ also as $e[2]$, and let E_{ij} denote the matrix satisfying $E_{ij}e[k]=\delta_{jk}e[i]$. Then, the R-matrix $R(z,\lambda)\in End(V)$ is given by

$$\begin{aligned} R(z,\lambda;\eta,\tau)=\quad & E_{11}\otimes E_{11}+E_{22}\otimes E_{22}+\alpha(z,\lambda)E_{11}\otimes E_{22} \\ &+\beta(z,\lambda)E_{12}\otimes E_{21}+\beta(z,-\lambda)E_{21}\otimes E_{12}+\alpha(z,-\lambda)E_{22}\otimes E_{11}\,, \end{aligned} \tag{18}$$

where $h=E_{11}-E_{22}$ and $\alpha(z,\lambda)$ and $\beta(z,\lambda)$ are defined by

$$\alpha(z,\lambda)=\frac{\theta(z)\theta(\lambda+2\eta)}{\theta(z-2\eta)\theta(\lambda)}\,,\qquad \beta(z,\lambda)=-\frac{\theta(z+\lambda)\theta(2\eta)}{\theta(z-2\eta)\theta(\lambda)}\,. \tag{19}$$

The theta function has been given by

$$\theta(z;\tau)=2p^{1/4}\sin\pi z\prod_{n=1}^{\infty}(1-p^{2n})(1-p^{2n}\exp(2\pi iz))(1-p^{2n}\exp(-2\pi iz))\,, \tag{20}$$

where the nome p is related to the parameter τ by $p=\exp(\pi i\tau)$ with $\mathrm{Im}\quad \tau>0$.

Let us now review the construction of the eigenvectors of the elliptic algebra $E_{\tau,\eta}(sl_2)$ at the discrete coupling parameter: $2N\eta=m_1+m_2\tau$, where N, m_1 and m_2 are any given integers.[5] Here we note that $2N\eta=m_1+m_2\tau$ corresponds to $2N\eta=2m_1K+im_2K$ in (4). Hereafter we assume $m_2=0$ for simplicity. Let $W=V(z_1)\otimes\cdots\otimes V(z_L)$ be the Lth tensor product of the evaluation modules $V_{\Lambda_j}(z_j)$'s with $\Lambda_j=1$ for all j.[17,18] The transfer matrix $T(z)$ of $E_{\tau,\eta}(sl_2)$ is given by the trace of the L-operator acting on the module W

$$\begin{aligned} L(z,\lambda)=\ & R^{(01)}(z-z_1,\lambda-2\eta\sum_{j=2}^{L}h^{(j)}) \\ &\times R^{(02)}(z-z_2,\lambda-2\eta\sum_{j=3}^{L}h^{(j)})\cdots R^{(0L)}(z-z_L,\lambda) \end{aligned} \tag{21}$$

Let us consider the mth product of the creation operators $b(t_j)$'s on the vacuum.[18,21] Let us assume the number m satisfies the following condition

$$2m=L-rN\,,\quad \text{for}\quad r\in\mathbf{Z} \tag{22}$$

Hereafter we also assume that rm_1 is even. We introduce a function $g_c(\lambda)$ by $g_c(\lambda) = e^{c\lambda}\prod_{j=1}^{m}(\theta(\lambda-2\eta j)/\theta(2\eta))$. Vector v_c is defined by $v_c = g_c(\lambda)v_0$, where v_0 is the highest weight vector of W: $hv_0 = Lv_0$. Then, making use of the fundamental commutation relations[18] associated with $b(z_j)$'s, we can show that $b(t_1)\cdots b(t_m)v_c$ is an eigenvector of the transfer matrix $T(z)$ with the eigenvalue $C_0(z)$

$$C_0(w) = e^{-2\eta c}\prod_{j=1}^{m}\frac{\theta(w-t_j+2\eta)}{\theta(w-t_j)} + e^{2\eta c}\prod_{j=1}^{m}\frac{\theta(w-t_j-2\eta)}{\theta(w-t_j)}\prod_{\alpha=1}^{L}\frac{\theta(w-z_\alpha)}{\theta(w-z_\alpha-2\eta)}, \tag{23}$$

if rapidities $t_1, t_2, \ldots, t_m$ satisfy the Bethe ansatz equations

$$\prod_{k=1}^{L}\frac{\theta(t_j-p_k)}{\theta(t_j-q_k)} = e^{-4\eta c}\prod_{k=1;k\neq j}^{m}\frac{\theta(t_j-t_k+2\eta)}{\theta(t_j-t_k-2\eta)}, \qquad \text{for} \quad j=1,\ldots,m. \tag{24}$$

The vector $b(t_1)\cdots b(t_m)v_c$ is explicitly given by the following:[18]

$$(-1)^m\, e^{c(\lambda+2\eta m)}\sum_{P\in\mathcal{S}_m}\sum_{1\le j_1<\cdots<j_m\le L}\prod_{\alpha=1}^{m}\prod_{\beta=j_\alpha+1}^{L}\frac{\theta(t_{P\alpha}-z_\beta)}{\theta(t_{P\alpha}-z_\beta-2\eta)}$$
$$\times\prod_{1\le\alpha<\beta\le m}f_{P\alpha P\beta}\prod_{\alpha=1}^{m}\frac{\theta(\lambda+t_{P\alpha}-z_{j_\alpha}-2\eta(rN-j_\alpha+\alpha))}{\theta(t_{P\alpha}-z_{j_\alpha}-2\eta)}\,\sigma_{j_1}^{-}\cdots\sigma_{j_m}^{-}|0\rangle \tag{25}$$

Here σ_j^- denotes the Pauli matrix σ^- acting on the jth site, $\mathcal{S}$ the symmetric group, $|0\rangle$ the vacuum vector and $f_{jk} = \theta(t_j-t_k-2\eta)/\theta(t_j-t_k)$.

4. The eigenvectors of the 8V CSOS model

Let us replace λ with $\lambda+\lambda_0$ in the L-operator (21) on W. Here λ_0 is independent of λ. Then, the R-matrix $R(z,\lambda+\lambda_0)$ is related to the Boltzmann weights $w(a,b,c,d;z,\lambda_0)$ of the 8V CSOS model through the following relation

$$R(z,-2\eta d+\lambda_0)e[c-d]\otimes e[b-c] = \sum_a w(a,b,c,d;z,\lambda_0)e[b-a]\otimes e[a-d] \tag{26}$$

Here a,b,c,d denote the spin variables of the IRF (the Interaction Round a Face) model which take integer values.[10] The spin variables have the constraint that the difference between the values of two nearest-neighboring spins should be given by ±1. Furthermore, for the 8V CSOS model discussed in Refs.,[11–13] the spin variables take the restricted values such as $0, 1, \ldots, N-1$ where the values 0 and $N-1$ can be assigned for adjacent spins.

Through the relation (26), we can show that the transfer matrix $T(z)$ of $E_{\tau,\eta}(sl_2)$ acting on the "path basis" corresponds to that of the 8V CSOS model.[10,18] Here we note that a "path" is given by a sequence of spin values satisfying the constraints on adjacent spins. Explicitly we consider the following[18]

$$|a_1,a_2,\cdots a_L\rangle(\lambda) = \delta(\lambda+2\eta a_1)\,e[a_1-a_2]\otimes e[a_2-a_3]\otimes\cdots\otimes e[a_L-a_1] \tag{27}$$

Here for the 8V CSOS model, we assume that $a_L - a_1 \equiv \pm 1 \pmod{N}$. Expressing the eigenvector $b(t_1)\cdots b(t_m)v_c$ of $T(z)$ in terms of the path basis, we obtain that of the transfer matrix of the 8V CSOS model.

5. The degenerate eigenvectors of the transfer matrix of the 8V CSOS model

Let us now assume that out of m rapidities $t_1,\ldots,t_m$, the first R rapidities t_j for $j=1,\ldots,R$ are of standard ones satisfying the Bethe ansatz equations (24) with m replaced by R, while the remaining NF rapidities are formal solutions given by

$$t_{(\alpha,j)} = t_{(\alpha)} + \eta(2j-N-1) + \epsilon\, r_j^{(\alpha)}\,, \qquad \text{for} \quad j=1,\ldots,N\,. \tag{28}$$

We call the set of N rapidities $t_{(\alpha,1)},\ldots,t_{(\alpha,N)}$, the complete N-string with center $t_{(\alpha)}$. Here the index α runs from 1 to F. Furthermore, we assume that the index (α,j) corresponds to the number $R+N(\alpha-1)+j$ for $1\le\alpha\le F$ and $1\le j\le N$. We note that the complete strings were suggested in Ref.[10] in another context.

Using the fundamental commutation relations, we can show when $\epsilon\neq 0$

$$\begin{aligned} T(z)b(t_1)\cdots b(t_{R+NF})\,v_c &= C_0(z)b(t_1)\cdots b(t_{R+NF})\,v_c \\ &+ \left(\sum_{j=1}^{R} + \sum_{j=R+1}^{R+NF}\right) C_j\, b(t_1)\cdots b(t_{j-1})b(z)b(t_{j+1})\cdots b(t_{R+NF})\,v_c\,. \end{aligned} \tag{29}$$

We divide eq. (29) by ϵ, and send ϵ to zero. Then, we can show that each of the terms of eq. (29) indeed converges, by making use of the following formula

$$\prod_{1\le\alpha<\beta\le m} f_{P\alpha P\beta} = \prod_{1\le\alpha<\beta\le m} f_{\alpha\beta} \times \prod_{1\le j<k\le m} \left(\frac{\theta(t_j-t_k+2\eta)}{\theta(t_j-t_k-2\eta)}\right)^{H(P^{-1}j-P^{-1}k)}, \tag{30}$$

for $P\in\mathcal{S}_m$. Here $H(x)$ denotes the Heaviside step function: $H(x)=1$ for $x>0$, $H(x)=0$ otherwise. The symbol $P\in\mathcal{S}_m$ denotes an element P of the symmetric group of m elements, where j is sent to $Pj\in\{1,2,\ldots,m\}$ for $j=1,\ldots m$. The formula (30) has been proven in Ref..[22]

Let us consider the following function of variable z[5]

$$\begin{aligned} G(z) = \sum_{a=1}^{N} e^{-4\eta c a} \prod_{j=a+1}^{N}\prod_{k=1}^{R} \frac{\theta(z-t_k+\eta(2j-N+1))}{\theta(z-t_k+\eta(2j-N-3))} \\ \times \prod_{\beta=1}^{L} \frac{\theta(z-z_\beta+\eta(2j-N-3))}{\theta(z-z_\beta+\eta(2j-N-1))} \end{aligned} \tag{31}$$

Hereafter we assume $\exp(4N\eta c)=1$. Then, the centers $t_{(\alpha)}$'s are determined by

$$G(z=t_{(\alpha)})=0\,, \quad \text{for} \quad \alpha=1,\ldots,F\,. \tag{32}$$

We can show that the zeros of (32) also form complete N strings, and also that the number of zeros of (32) is given by $L-2R$, by using the Bethe ansatz equations

(24).[5] Thus, the number of independent solutions to (32) is given by $(L-2R)/N$, which leads to the dimension $2^{(L-2R)/N}$ through the binomial expansion. Thus, for the transfer matrix of the 8V CSOS model, any standard Bethe ansatz eigenvector with R rapidities has the degeneracy of $2^{(L-2R)/N}$.

Let us now consider the connection of the CSOS model to the six-vertex model. Taking the trigonometric limit: $\tau \to i\infty$ and sending λ_0 to infinity with some gauge transformations, the L-operator of the 8V CSOS model becomes that of the six-vertex model. We may assume that the trigonometric limits of the R rapidities of the Bethe ansatz equations (24) with $\exp(4\eta c) = 1$ satisfy the trigonometric Bethe ansatz equations of the six-vertex model. Then, the degenerate eigenvectors with F complete N strings for the 8V CSOS model become those of the six-vertex model with F complete N strings. Thus, we have shown that the corresponding degenerate eigenspace is spanned by the eigenvectors having complete N strings, and also that the dimension is given by $2^{(L-2R)/N} = 2^{2S^Z_{max}/N}$ since the highest weight S^Z_{max} is given by $L/2 - R$. The result should be consistent with the previous studies.[1–4]

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THE CHERN–SIMONS INVARIANT IN THE BERRY PHASE OF A TWO BY TWO HAMILTONIAN

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By varying (x, y, z) within a manifold $\mathcal{M}$, the positive (negative)-energy eigenvectors of the 2×2 Hamiltonian $H = x\sigma_x + y\sigma_y + z\sigma_z$ (where $\sigma_{x,y,z}$ are the Pauli matrices) form a U(1) fiber bundle. For certain $\mathcal{M}$ the bundle has non-trivial topology. For example when $\mathcal{M} = S^2$ the associated bundle has non-zero Chern number indicating that it is topologically non-trivial at the highest level. In this paper we construct a simple 2×2 Hamiltonian whose eigen-vector bundle exhibits a more subtle topological non-triviality when $\mathcal{M}$ is a closed three-manifold. This non-trivial topology is characterized by non-zero Chern-Simons invariant.

Twenty years ago I was a graduate student at M.I.T.. Together with many friends in the Boston area, I was often invited to Fred's house for Chinese holidays. I remember vividly that in one occasion Fred entertained us by balancing a women's slipper on his nose. Those parties meant a lot to a young men just arrived in a foreign country.

Academically Fred introduced me to the field of statistical mechanics. At that time my thesis work mainly dealt with the electronic structure of semiconductor surfaces. To be frank, I was a little bored with that subject. In a lucky incident I stumbled upon a problem concerning the magnetic properties of spin 1/2 antiferromagnet on a triangular lattice. (Until today this problem is still of considerable interest.) I decided to attack the classical version of this problem, and before long I further simplified the problem by assuming the spins only have two components.

It turns out that this problem (classical antiferromagnetic xy model on triangular lattice) is not trivial at all. Fred advised me to do a mean-field calculation first. I took the advice and soon discovered the ground state degeneracy is $2 \times \infty$. Here ∞ comes from the global spin rotation symmetry and 2 comes from an interesting chirality ordering in the ground state. Together with Fred (and another fellow graduate student and my thesis adviser) we wrote a paper on the mean-field phase diagram of this problem. This work initiated a series of further studies of the critical behavior when the spins become disordered.

Recently Fred get interested in the knot theory in statistical mechanics. As to myself, I have been working in the field of quantum statistical mechanics of strongly

correlated systems. One day, while teaching the Berry phase to my quantum mechanics class, I encounter the following question. It is well known that the Berry phase of a spin 1/2 in external magnetic field (Eq. (1)) is the integral of the vector potential due to a magnetic monopole. In mathematical language this vector potential is the connection on a non-trivial fiber bundle. The monopole strength is the Chern number. I want to know whether we can define a simple 2×2 Hamiltonian whose Berry connection shows zero Chern number but non-zero Chern-Simons invariant.

Since Fred's Festschrift takes place in the Nankai Institute which was founded by professor Chern (to whom I have tremendous admiration) I thought this subject is particularly appropriate.

Since it's discovery in 1984,[1] the Berry phase has played an important role in quantum mechanics. For a simple example of the Berry phase, consider the following two by two Hamiltonian

$$H(\mathbf{r}) = x\sigma_x + y\sigma_y + z\sigma_z \equiv \mathbf{r} \cdot \vec{\sigma}, \tag{1}$$

where $\sigma_{x,y,z}$ are the three components of the Pauli matrices and x, y, z are real parameters. For a fixed $\mathbf{r} = (x, y, z)$ the Hamiltonian in Eq. (1) has two eigenvalues $E_\pm = \pm\sqrt{x^2 + y^2 + z^2} \equiv \pm|\mathbf{r}|$.

Let us focus on the eigenvector $|\psi(\mathbf{r})>$ associated with the positive eigenvalue for the rest of the paper. The Berry phase induced by an adiabatic evolution of $\mathbf{r}$ around a closed loop C is given by

$$\gamma = \oint_C dx^\mu A^b_\mu(\mathbf{r}), \tag{2}$$

where

$$A^b_\mu(\mathbf{r}) = \frac{1}{i} < \psi(\mathbf{r})|\partial_\mu \psi(\mathbf{r}) > . \tag{3}$$

It turns out that the A^b_μ in Eq. (3) has a geometric interpretation as we shall explain in the following.

At a fixed $\mathbf{r}$ if one is given a spinor (i.e. a 2-component column vector) satisfying

$$\begin{aligned} &H(\mathbf{r})|\psi(\mathbf{r}) >= |\mathbf{r}||\psi(\mathbf{r}) > \\ &< \psi(\mathbf{r})|\psi(\mathbf{r} >= 1, \end{aligned} \tag{4}$$

one can generate a continuum of other spinors which satisfy Eq. (4) by the transformation

$$|\psi(\mathbf{r}) > \rightarrow e^{i\theta}|\psi(\mathbf{r}) > . \tag{5}$$

This family of $|\psi(\mathbf{r}) >$ spans an internal space that is invariant under a U(1) group of transformations (Eq. (5)). As $\mathbf{r}$ varies through a manifold $\mathcal{M}$ (henceforth referred as the base space) the internal space sweeps out a geometric object called "fiber bundle". Since U(1) leaves the internal space invariant this fiber bundle is a U(1)

bundle. In the following we shall refer to such fiber bundle as the eigenbundle of Eq. (1).

It turns out that Eq. (3) is precisely the geometric connection on the eigenbundle.[2] The connection A_μ^b defined above is not unique. Indeed, by performing the transformation $|\psi(\mathbf{r})> \rightarrow e^{i\theta(\mathbf{r})}|\psi(\mathbf{r})>$ we induce a "gauge transformation" on A_μ^b:

$$A_\mu^b \rightarrow A_\mu^b + \partial_\mu \theta. \tag{6}$$

It is obvious that the Berry phase (Eq. (2)) is gauge invariant.

Next we shall focus on two-dimensional base spaces $\mathcal{M}$ that are closed surfaces. It turns out that if $\mathcal{M}$ encloses the origin (for example $\mathcal{M} = S^2 = \{\mathbf{r};\ |\mathbf{r}| = 1\}$), it is impossible to choose a gauge in which A_μ^b is non-singular everywhere. In order to obtain locally non-singular A_μ^b it is necessary to divide $\mathcal{M}$ into a number of (overlapping) patches so that 1) A_μ^b is non-singular in each patch, and 2) in the region where two patches overlap the different A_μ^b's differ only by a gauge transformation. Historically this problem was encountered by Dirac when he tried to write down the vector potential in the neighborhood of a magnetic monopole.[3] It turns out that under the framework of quantum mechanics, condition 2) requires the strength of the monopole to be quantized.[3,4]

In geometry it is known that the non-existence of an everywhere-nonsingular connection is the manifestation of non-trivial topology. In his seminal work S.S. Chern discovered a set of invariants to characterize such non-triviality.[5] For the simple case we are considering the invariants reduce to a single number, the Chern number:

$$\mathcal{C} = \frac{1}{4\pi} \int_{\mathcal{M}} d^2x \epsilon^{\mu\nu} F_{\mu\nu}^b. \tag{7}$$

Here $F_{\mu\nu}^b = \partial_\mu A_\nu^b - \partial_\nu A_\mu^b$ is the curvature associated with A_μ^b. For the eigenbundle of Eq. (1) it is simple to show that $\mathcal{C} = 1/2$ or 0 depending on whether $\mathcal{M}$ encloses the origin. If we interpret $F_{\mu\nu}$ as magnetic field, the above result suggests that $\mathcal{C}$ is the total magnetic flux (through $\mathcal{M}$) produced by a magnetic monopole located at $\mathbf{r} = 0$. In a proof similar but more general than that given in Ref. 3, Chern showed that $\mathcal{C}$ should be quantized to values $n/2$ where $n =$ integer.[5]

Since $\mathcal{C} > 1/2$ is allowed, it is interesting to ask what kind of Hamiltonian will have $\mathcal{C} = n/2$ $(n > 1)$ eigenbundles. One answer is given by the following $(n+1) \times (n+1)$ matrix

$$H(\mathbf{r}) = \mathbf{r} \cdot \mathbf{S}, \tag{8}$$

where $\mathbf{S} = (S_x, S_y, S_z)$ are the matrices representing the three generators of SU(2) in the spin $S = n/2$ representation. For example for $n = 2$ we have

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{9}$$

There is another way of modifying Eq. (1) which also leads to $\mathcal{C} = n/2$ eigenbundle. Interestingly this time we do not need to enlarge the dimension of H. Consider the following 2×2 matrix

$$H(\mathbf{r}) = \hat{h}(\mathbf{r}) \cdot \vec{\sigma}, \tag{10}$$

where $\hat{h}(\mathbf{r})$ is a suitably chosen unit vector field that defines a mapping from $\mathcal{M}$ (a closed two-manifold) to S^2. It is known that such mappings can be classified into homotopy classes each labeled by an integer

$$\mathcal{P} = \int_{\mathcal{M}} d^2x \epsilon^{\mu\nu} J_{\mu\nu}. \tag{11}$$

Here the Pontryagin form $J_{\mu\nu}$ is given by

$$J_{\mu\nu} = \frac{1}{4\pi} \hat{h} \cdot (\partial_\mu \hat{h} \times \partial_\nu \hat{h}). \tag{12}$$

We will later show that by choosing a $\hat{h}(\mathbf{r})$ with $\mathcal{P} = n$ the eigenbundle of Eq. (10) exhibits $\mathcal{C} = n/2$.

The Chern number records the highest level of topological non-triviality. When the Chern number vanishes the eigenbundle can still be non-trivial at a more subtle level. Let us consider closed three-manifold in which $\mathcal{C} = 0$ for all closed sub two-manifolds, which implies the absence of monopole. Without monopole the "flux lines" associated with the vector field $\epsilon^{\mu\nu\lambda} F_{\nu\lambda}$ form closed loops. Under this condition there is a topological interesting situation in which these flux lines link with one another. It is clear that this class of eigenbundles are topologically distinct from those without linking flux lines.

In 1974 Chern and Simons discovered an invariant, the Chern-Simons invariant, that quantifies this more subtle level of topological non-triviality.[6] For a closed three-manifold $\mathcal{M}$ the Chern-Simons invariant is given by

$$\mathcal{CS} = \frac{1}{4\pi} \int_{\mathcal{M}} d^3x \epsilon^{\mu\nu\lambda} A^b_\mu \partial_\nu A^b_\lambda. \tag{13}$$

We note that when $\mathcal{M}$ is closed $\mathcal{CS}$ is gauge invariant. The topological information recorded by $\mathcal{CS}$ is precisely the linking between the flux lines. The fact that linking is only defined in three dimensions explains why the Chern-Simons invariant requires three dimensional base space.

Since there is another level of topological non-triviality, it is natural to ask whether one can modify Eq. (1) so that the eigenbundle exhibits such topology. We shall prove that the Hamiltonian given by Eq. (10) also works so long as $\hat{h}(\mathbf{r})$ is chosen appropriately.

Now let us restrict ourselves to the case where the base space $\mathcal{M}$ is a simply connected closed three-manifold and $\mathbf{r}$ labels the points in it. In such case $\hat{h}(\mathbf{r})$ is a

mapping from a simply-connected closed three-manifold to S^2. The work of Hopf shows that such mapping can also be classified into homotopy classes by an integral valued Hopf invariant $\mathcal{H}$. A Hopf map is a smooth configuration of $\hat{h}(\mathbf{r})$. Due to the fact that there are only two linearly independent directions for $\partial_\mu \hat{h}$, it follows

$$\epsilon^{\mu\nu\lambda}\partial_\mu J_{\nu\lambda} = 0. \tag{14}$$

As the result the flow lines associated with $\epsilon^{\mu\nu\lambda}J_{\nu\lambda}$ form closed loops. For a non-trivial Hopf map any pair of J-loops link with each other. Because of Eq. (14) there exists a A^h_μ so that

$$J_{\mu\nu} = \frac{1}{4\pi}(\partial_\mu A^h_\nu - \partial_\nu A^h_\mu). \tag{15}$$

The Hopf invariant is simply the Chern-Simons invariant for A^h_μ,[7,8] i.e.,

$$\mathcal{H} = \frac{1}{4\pi}\int_{\mathcal{M}} d^3x \epsilon^{\mu\nu\lambda} A^h_\mu \partial_\nu A^h_\lambda. \tag{16}$$

In the rest of the paper we prove the following.

$$\begin{aligned}&(i)\ \text{For closed two} - \text{manifold } \mathcal{M} \text{ the eigen bundle of}\\ &Eq.\ (10) \text{ has } \mathcal{C} = n/2 \text{ if } \hat{h}(\mathbf{r}) \text{ has } \mathcal{P} = n.\\ &(ii)\ \text{For closed three} - \text{manifold } \mathcal{M} \text{ the eigen bundle of}\\ &Eq.\ (10) \text{ has } \mathcal{CS} = n \text{ if } \hat{h}(\mathbf{r}) \text{ has } \mathcal{H} = n.\end{aligned} \tag{17}$$

The proof amounts to show the following identity

$$F^b_{\mu\nu} = 4\pi J_{\mu\nu}. \tag{18}$$

The Berry curvature is given by

$$\begin{aligned}F^b_{\mu\nu} &= \frac{1}{i}[\partial_\mu < \psi_+|\partial_\nu\psi_+ > -\partial_\nu < \psi_+|\partial_\mu\psi_+ >]\\ &= \frac{1}{i}[< \partial_\mu\psi_+|\partial_\nu\psi_+ > - < \partial_\nu\psi_+|\partial_\mu\psi_+ >].\end{aligned} \tag{19}$$

To compute $< \partial_\mu\psi_+|\partial_\nu\psi_+ > - < \partial_\nu\psi_+|\partial_\mu\psi_+ >$ we insert a complete set of states ($I = \sum_{n=\pm} |\psi_n >< \psi_n|$), and that gives

$$\begin{aligned}&< \partial_\mu\psi_+|\partial_\nu\psi_+ > - < \partial_\nu\psi_+|\partial_\mu\psi_+ >\\ &= \sum_{n=\pm} < \partial_\mu\psi_+|\psi_n >< \psi_n|\partial_\nu\psi_+ > -[\mu \leftrightarrow \nu]\\ &=< \partial_\mu\psi_+|\psi_- >< \psi_-|\partial_\nu\psi_+ > -[\mu \leftrightarrow \nu].\end{aligned} \tag{20}$$

In reaching the last line we have used the fact that $< \partial_\mu\psi_+|\psi_+ >< \psi_+|\partial_\nu\psi_+ > -[\mu \leftrightarrow \nu] = 0$.

To compute $< \psi_-|\partial_\nu\psi_+ >$ in Eq. (20) we express the eigenvector of $H(\mathbf{r}') = H(\mathbf{r}+\delta\mathbf{r}) = H(\mathbf{r}) + \delta x^\lambda \partial_\lambda H$ in terms of those of $H(\mathbf{r})$ via first order perturbation

theory. Up to the first order in δx^μ, we obtain

$$|\psi_+(\mathbf{r}')> = \left[|\psi_+> + \frac{<\psi_-|\delta x^\lambda \partial_\lambda H|\psi_+>}{E_+ - E_-}|\psi_->\right]$$
$$= \left[|\psi_+> + \frac{<\psi_-|\delta x^\lambda \partial_\lambda \hat{h}\cdot\vec{\sigma}|\psi_+>}{2}|\psi_->\right]. \tag{21}$$

Eq. (21) implies that

$$<\psi_-|\partial_\nu\psi_+> = \frac{<\psi_-|\partial_\nu \hat{h}\cdot\vec{\sigma}|\psi_+>}{2}. \tag{22}$$

As the result we have

$$<\partial_\mu\psi_+|\psi_-><\psi_-|\partial_\nu\psi_+> - [\mu\leftrightarrow\nu]$$
$$= \frac{1}{4}[<\psi_+|\partial_\mu\hat{h}\cdot\vec{\sigma}|\psi_-><\psi_-|\partial_\nu\hat{h}\cdot\vec{\sigma}|\psi_+> - [\mu\leftrightarrow\nu]$$
$$= \frac{1}{4}\sum_{n=\pm}[<\psi_+|\partial_\mu\hat{h}\cdot\vec{\sigma}|\psi_n><\psi_n|\partial_\nu\hat{h}\cdot\vec{\sigma}|\psi_+> - [\mu\leftrightarrow\nu]$$
$$= \frac{1}{4}[<\psi_+|[\partial_\mu\hat{h}\cdot\vec{\sigma}, \partial_\nu\hat{h}\cdot\vec{\sigma}]|\psi_+>]$$
$$= \frac{i}{2}\epsilon_{abc}(\partial_\mu\hat{h}_a)(\partial_\nu\hat{h}_b)[<\psi_+|\sigma_c|\psi_+>]$$
$$= \frac{i}{2}\epsilon_{abc}(\partial_\mu\hat{h}_a)(\partial_\nu\hat{h}_b)\hat{h}_c = \frac{i}{2}\hat{h}\cdot(\partial_\mu\hat{h}\times\partial_\nu\hat{h}). \tag{23}$$

Going from the second to the third line of Eq. (23) we have used the fact that $<\psi_+|\partial_\mu\hat{h}\cdot\vec{\sigma}|\psi_+><\psi_+|\partial_\nu\hat{h}\cdot\vec{\sigma}|\psi_+> - [\mu\leftrightarrow\nu] = 0$. Substituting Eq. (20) and Eq. (23) into Eq. (19) we obtain

$$F^b_{\mu\nu} = \frac{1}{2}\hat{h}\cdot\partial_\mu\hat{h}\times\partial_\nu\hat{h} = 4\pi J_{\mu\nu}. \tag{24}$$

After establishing Eq. (18) it is simple to prove (i) and (ii) of (17). For (i) the Chern number is given by

$$\mathcal{C} = \frac{1}{4\pi}\int_{\mathcal{M}} d^2x\ \epsilon^{\mu\nu}F^b_{\mu\nu} = \frac{1}{2}\int_{\mathcal{M}} d^2x\ \epsilon^{\mu\nu}J_{\mu\nu} = \mathcal{P}/2. \tag{25}$$

Thus $\mathcal{P} = n$ implies $\mathcal{C} = n/2$. Now let us prove (ii) of (17). Eq. (15) and Eq. (18) imply that

$$F^b_{\mu\nu} = 4\pi J_{\mu\nu} = F^h_{\mu\nu}. \tag{26}$$

Thus A^h_μ and A^b_μ differ at most by a pure gauge

$$A^b_\mu = A^h_\mu + \partial_\mu\phi. \tag{27}$$

Since Eq. (13) is gauge invariant when $\mathcal{M}$ is a closed manifold we conclude that

$$\begin{aligned} \mathcal{CS} &= \frac{1}{8\pi}\int_{\mathcal{M}} d^3x\epsilon^{\mu\nu\lambda}A^b_\mu F^b_{\nu\lambda} = \frac{1}{8\pi}\int_{S^3} d^3x\epsilon^{\mu\nu\lambda}A^h_\mu F^h_{\nu\lambda} \\ &= \mathcal{H}. \end{aligned} \tag{28}$$

Thus $\mathcal{H} = n$ implies $\mathcal{CS} = n$.

In physics one often encounters the Berry phase when a system posses's both "fast" and "slow" dynamic degrees of freedom. When the fast degrees of freedom are integrated it often produces, as part of the effective action of the slow variables, a Berry phase term that is non-zero even when the slow variables change adiabatically with time. Such term can fundamentally alter the behavior of the slow variables.

Here we present an example where the fast degrees of freedom generate an effective action represented by the Hopf invariant of the slow variables. The model is a field theory in 2+1 space-time dimensions. It consists of two fields: 1) a fermion field $\psi_\sigma(\mathbf{r}, t)$, and 2) an unit vector field $\hat{n}(\mathbf{r}, t)$. The Lagrangian density is given as

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_\psi + \mathcal{L}_n - g\hat{n}\cdot\bar{\psi}_\alpha\vec{\sigma}_{\alpha\beta}\psi_\beta \\ \mathcal{L}_\psi &= \bar{\psi}_\alpha(\partial_t - \mu)\psi_\alpha - \frac{1}{2m}\bar{\psi}_\alpha(\nabla - i\mathbf{A}_{ex})^2\psi_\alpha \\ \mathcal{L}_n &= i\,\Omega[\hat{n}] + \frac{K}{2}|\nabla\hat{n}|^2. \end{aligned} \tag{29}$$

In the above m, g, c, μ are parameters of the model, $\mathbf{A}_{ex}$ is the vector potential of an external magnetic field B, i.e., $\partial_x A_y - \partial_y A_x = B$, and $\delta\Omega/\delta\hat{n} = \hat{n}\times\partial_t\hat{n}$. Physically $\mathcal{L}_\psi$ describes fermions moving in an external magnetic field, and $\mathcal{L}_n$ describes the dynamics of magnetic moments in a ferromagnet. The last term in the first equation is the Zeeman coupling between the fermions and the magnetic moments. By adjusting μ we can tune the density ($\bar{\rho}$) of the fermions so that

$$\bar{\rho} = k\frac{B}{\phi_0}, \tag{30}$$

where $\phi_0 = 2\pi$ is the Dirac flux quantum and k is an integer. When Eq. (30) is satisfied, the ground state of the fermions is an "integer-quantum Hall liquid".[9] Let us further assume that g is large so that locally the electron spins align with the direction of $\hat{n}$. Under that condition integrating out the fermion field produces a term $\frac{k}{4\pi}\int d^2xdt\epsilon^{\mu\nu\lambda}A^h_\mu\partial_\nu A^h_\lambda$, which is proportional to the Hopf invariant of the $\hat{n}(\mathbf{r}, t)$. This term has the effect of changing the spins and statistics of solitons (the skyrmions) in the $\hat{n}(\mathbf{r}, t)$ field.[7]

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MUTUALLY LOCAL FIELDS FROM FORM FACTORS*

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We compare two different methods of computing form factors. One is the well established procedure of solving the form factor consistency equations and the other is to represent the field content as well as the particle creation operators in terms of fermionic Fock operators. We compute the corresponding matrix elements for the complex free fermion and the Federbush model. The matrix elements only satisfy the form factor consistency equations involving anyonic factors of local commutativity when the corresponding operators are local. We carry out the ultraviolet limit, analyze the momentum space cluster properties and demonstrate how the Federbush model can be obtained from the $SU(3)_3$-homogeneous sine-Gordon model. We propose a new class of Lagrangians which constitute a generalization of the Federbush model in a Lie algebraic fashion. For these models we evaluate the associated scattering matrices from first principles, which can alternatively also be obtained in a certain limit of the homogeneous sine-Gordon models.

1. Introduction

One of the most central concepts in relativistic quantum field theory, like Einstein causality and Poincaré covariance, are captured in local field equations and commutation relations. In fact this principle is widely considered as so pivotal that it constitutes the base of a whole subject, i.e. local quantum physics (algebraic quantum field theory)[2] which takes the collection of all operators localized in a particular region generating a von Neumann algebra, as its very starting point.

On the other hand, in the formulation of a quantum field theory, one may alternatively start from a particle picture and investigate the corresponding scattering theories. In particular for 1+1 dimensional integrable quantum field theories this latter approach has been proved to be impressively successful. As its most powerful tool one exploits here first the bootstrap principle,[3–5] which allows to write down exact, i.e. non-perturbative, scattering matrices. Ignoring subtleties of non-asymptotic states, it is essentially possible to obtain the particle picture from the field formulation by means of the LSZ-reduction formalism.[6] However, the question

*based on reference 1

of how to reconstruct the field content, or at least part of it, from the scattering theory is in general still an outstanding issue.

This talk is also devoted to this question in the sense that we provide explicit expressions for operators $\mathcal{O}(x)$ located at x in terms of fermionic Fock fields. Particular emphasis is put on the question whether these operators are really local in the sense that they (anti)-commute for space-like separations with themselves,

$$[\mathcal{O}(x), \mathcal{O}(y)] = 0 \qquad \text{for} \qquad (x-y)^2 < 0 \tag{1}$$

and how this property is reflected in the form factor consistency equations. It will turn out that from possible matrix elements the form factor consistency equations select out those which correspond to mutually local operators. We argue that the presence of the factor of local commutativity in these equations is absolutely essential.

2. Determination of form factors

Let us assume that there is no backscattering in our model and that we have explicitly determined its two-particle scattering matrix, which can be expressed as a phase in this case. We further presume that the S-matrix results from braiding two particle creation operators $Z_\mu^\dagger(\theta)$ for stable particles of type μ with rapidity θ, which obey the Faddeev-Zamolodchikov algebra[7]

$$Z_i^\dagger(\theta_i) Z_j^\dagger(\theta_j) = S_{ij}(\theta_{ij}) Z_j^\dagger(\theta_j) Z_i^\dagger(\theta_i) = \exp[2\pi i \delta_{ij}(\theta_{ij})] Z_j^\dagger(\theta_j) Z_i^\dagger(\theta_i)\,. \tag{2}$$

As common we parameterize the two-momentum $\vec{p}$ by the rapidity variable θ as $\vec{p} = m(\cosh\theta, \sinh\theta)$ and abbreviate $\theta_{ij} := \theta_i - \theta_j$. In order to pass from scattering theory to fields, we want to determine the form factors, i.e. the matrix element of a local operator $\mathcal{O}(x)$ located at the origin between a multi-particle in-state and the vacuum

$$F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1,\ldots,\theta_n) \equiv \left\langle \mathcal{O}(0)\, Z_{\mu_1}^\dagger(\theta_1),\ldots, Z_{\mu_n}^\dagger(\theta_n) \right\rangle_{\text{in}}\,. \tag{3}$$

We distinguish between the mere matrix element $\tilde{F}_n^{\mathcal{O}}$ and the particular ones which also solve the consistency equations in 2.1, in which case we denote them as $F_n^{\mathcal{O}}$.

2.1. *Form factors from consistency equations*

Various schemes have been suggested to compute the objects in equation (3). One of them consists of solving a system of consistency equations which have to hold for the n-particle form factors based on some natural physical assumptions, like unitarity, crossing and bootstrap fusing properties[8–11]

$$F_n^{\mathcal{O}|\cdots\mu_i\mu_j\cdots}(\ldots,\theta_i,\theta_j,\ldots) = F_n^{\mathcal{O}|\cdots\mu_j\mu_i,\cdots}(\ldots,\theta_j,\theta_i,\ldots) S_{\mu_i\mu_j}(\theta_{ij})\,, \tag{4}$$

$$F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1+2\pi i,\ldots,\theta_n) = \gamma_{\mu_1}^{\mathcal{O}}\, F_n^{\mathcal{O}|\mu_2\cdots\mu_n\mu_1}(\theta_2,\ldots,\theta_n,\theta_1)\,, \tag{5}$$

$$F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1+\lambda,\ldots,\theta_n+\lambda) = e^{s\lambda} F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1,\ldots,\theta_n)\,, \tag{6}$$

$$\operatorname*{Res}_{\bar\theta\to\theta_0} F_{n+2}^{\mathcal{O}|\bar\mu\mu\mu_1\dots\mu_n}(\bar\theta+i\pi,\theta_0,\theta_1\dots\theta_n) = i(1-\gamma_\mu^{\mathcal{O}}\prod_{l=1}^{n} S_{\mu\mu_l}(\theta_{0l}))F_n^{\mathcal{O}|\mu_1\dots\mu_n}(\theta_1\dots\theta_n). \quad (7)$$

Here s is the Lorentz spin of the operator $\mathcal{O}$ and λ is an arbitrary real number. We omitted here the so-called bound state residue equation, which relates an $(n+1)$- to an n-particle form factor, since it will be of no importance to the explicit models we consider. We stress the importance of the constant $\gamma_\mu^{\mathcal{O}}$, the factor of so-called local commutativity defined through the equal time exchange relation of the local operator $\mathcal{O}(x)$ and the field $\mathcal{O}_\mu(y)$ associated to particle creation operators $Z_\mu^\dagger$

$$\mathcal{O}_\mu(x)\mathcal{O}(y) = \gamma_\mu^{\mathcal{O}}\,\mathcal{O}(y)\,\mathcal{O}_\mu(x) \qquad \text{for} \quad x^1 > y^1\,, \quad (8)$$

with $x^\mu = (x^0, x^1)$. This factor carries properties of the operator and not just of the $Z^\dagger$'s. An immediate consequence of its presence is that a frequently made statement has to be revised, namely, that (4)-(7) constitute operator independent equations, which require as the only input the two-particle scattering matrix. Here we demonstrate that apart from ± 1, which already occur in the literature, this factor can be a non-trivial phase. Thus the form factor consistency equations contain also explicitly non-trivial properties of the operators. To solve these equations at least for the lowest n-particle form factors is a fairly well established procedure, but it still remains a challenge to find closed analytic solutions for all n-particle form factors.

2.2. *Direct computation of matrix elements*

The most direct way to compute the matrix elements in (3) is to find explicit representations for the operators $Z_\mu^\dagger(\theta)$ and $\mathcal{O}(x)$. To represent the former operator is known in complete generality for theories not involving backscattering. A representation for these operators in the bosonic Fock space was first provided in Ref. 12

$$Z_i^\dagger(\theta) = \exp\left[-i\int_\theta^\infty d\theta'\,\delta_{il}(\theta-\theta')a_l^\dagger(\theta')a_l(\theta')\right] a_i^\dagger(\theta)\,. \quad (9)$$

where the a's satisfy the usual fermionic anti-commutation relations

$$\{a_i(\theta), a_j(\theta')\} = 0 \qquad \text{and} \qquad \{a_i(\theta), a_j^\dagger(\theta')\} = 2\pi\delta_{ij}\delta(\theta-\theta')\,. \quad (10)$$

Having obtained a fairly simple realization for the Z-operators, we may now seek to represent the operator content of the theory in the same Fock space. Hitherto, it is not known how to do this in general and we have to resort to a study of explicit models at this stage.

3. Complex free Fermions

Let us consider N complex (Dirac) free Fermions described as usual by the Lagrangian density

$$\mathcal{L}_{\rm FF} = \sum\nolimits_{\alpha=1}^{N} \bar\psi_\alpha(i\gamma^\mu\partial_\mu - m_\alpha)\psi_\alpha\,. \quad (11)$$

We define a prototype auxiliary field

$$\chi^{\alpha}_{\kappa}(x) = \int \frac{d\theta d\theta'}{4\pi^2} \Big[\kappa^{\alpha}(\theta,\theta') \left(a^{\dagger}_{\alpha}(\theta)a^{\dagger}_{\bar{\alpha}}(\theta')e^{i(p+p')\cdot x} + a_{\alpha}(\theta)a_{\bar{\alpha}}(\theta')e^{-i(p+p')\cdot x}\right) + \kappa^{\alpha}(\theta,\theta'-i\pi)\left(a^{\dagger}_{\bar{\alpha}}(\theta)a_{\bar{\alpha}}(\theta')e^{i(p-p')\cdot x} - a_{\alpha}(\theta)a^{\dagger}_{\alpha}(\theta')e^{-i(p-p')\cdot x}\right)\Big] \quad (12)$$

and intend to compute the matrix element of general operators composed out of these fields

$$\mathcal{O}^{\chi^{\alpha}_{\kappa}}(x) = :e^{\chi^{\alpha}_{\kappa}(x)}:,\ \hat{\mathcal{O}}^{\chi^{\alpha}_{\kappa}}(x) = :\int \frac{dp^1_{\alpha}}{2\pi p^0_{\alpha}}(a_{\alpha}(p)e^{-ip_{\alpha}\cdot x} + a^{\dagger}_{\bar{\alpha}}(p)e^{ip_{\alpha}\cdot x})e^{\chi^{\alpha}_{\kappa}(x)}:. \quad (13)$$

Employing Wick's first theorem, we compute[1]

$$\tilde{F}^{\mathcal{O}^{\chi^{\alpha}_{\kappa}}|n\times\bar{\alpha}\alpha}_{2n}(\theta_1,\ldots,\theta_{2n}) = \int \frac{d\theta'_1\ldots d\theta'_{2n}}{n!}\prod_{i=1}^{n}\kappa^{\alpha}(\theta'_{2i-1},\theta'_{2i})\det\mathcal{D}^{2n}\ , \quad (14)$$

$$\tilde{F}^{\hat{\mathcal{O}}^{\chi^{\alpha}_{\kappa}}|\alpha,n\times\bar{\alpha}\alpha}_{2n+1}(\theta_1,\ldots,\theta_{2n+1}) = \int \frac{d\theta'_1\ldots d\theta'_{2n+1}}{n!}\prod_{i=1}^{n}\kappa^{\alpha}(\theta'_{2i},\theta'_{2i+1})\det\mathcal{D}^{2n+1}, \quad (15)$$

where $\mathcal{D}^{\ell}$ is a rank ℓ matrix whose entries are given by

$$\mathcal{D}^{\ell}_{ij} = \cos^2[(i-j)\pi/2]\delta(\theta'_i-\theta_j)\,, \qquad 1\leq i,j\leq \ell\,. \quad (16)$$

Note that $\mathcal{O}^{\chi^{\alpha}_{\kappa}}(x)$ and $\hat{\mathcal{O}}^{\chi^{\alpha}_{\kappa}}(x)$ are in general non-local operators in the sense of (1). At the same time $\tilde{F}^{\mathcal{O}}_{n}$ is just the matrix element as defined on the r.h.s. of (3) and not yet a form factor of a local field, in the sense that it satisfies the consistency equations (4)-(7), which imply locality of $\mathcal{O}$. A rigorous proof of this latter implication to hold in generality is still an open issue. Let us now specify the function κ. The free fermionic theory possesses some very distinct fields, namely the disorder and order fields

$$\mu_{\alpha}(x) = :e^{\omega_{\alpha}(x)}: \qquad \text{and} \qquad \sigma_{\alpha}(x) = :\hat{\psi}_{\alpha}(x)\mu_{\alpha}(x):, \quad \alpha = 1,2, \quad (17)$$

respectively. We introduced here the fields

$$\omega_{\alpha}(x) = \chi^{\alpha}_{\kappa}(x), \quad \kappa^1(\theta,\theta') = -\kappa^2(-\theta,-\theta') = \frac{i}{2}\frac{e^{-\frac{1}{2}(\theta-\theta')}}{\cosh\frac{1}{2}(\theta-\theta')}\ . \quad (18)$$

We compute[1] the integrals in (14) and (15) for this case and obtained a closed expression for the n-particle form factors of the disorder and order operators

$$\begin{aligned} F^{\mu_1|n\times\bar{1}1}_{2n}(\theta_1,\ldots,\theta_{2n}) &= (-1)^n F^{\mu_2|n\times\bar{2}2}_{2n}(-\theta_1,\ldots,-\theta_{2n}) \\ F^{\mu_{\bar{1}}|n\times\bar{1}1}_{2n}(-\theta_1,\ldots,-\theta_{2n}) &= (-1)^n F^{\mu_{\bar{2}}|n\times\bar{2}2}_{2n}(\theta_1,\ldots,\theta_{2n}) \\ &= i^n 2^{n-1}\sigma_n(\bar{x}_1,\bar{x}_3,\ldots,\bar{x}_{2n-1})\mathcal{B}_{n,n}\,, \quad (19) \\ F^{\sigma_1|1(n\times\bar{1}1)}_{2n+1}(\theta_1,\ldots,\theta_{2n+1}) &= (-1)^n F^{\sigma_2|2(n\times\bar{2}2)}_{2n+1}(-\theta_1,\ldots,-\theta_{2n+1}) \\ F^{\sigma_{\bar{1}}|1(n\times\bar{1}1)}_{2n+1}(-\theta_1,\ldots,-\theta_{2n+1}) &= (-1)^n F^{\sigma_{\bar{2}}|2(n\times\bar{2}2)}_{2n+1}(\theta_1,\ldots,\theta_{2n+1}) \\ &= i^n 2^{n-1}\sigma_n(\bar{x}_1,\ldots,\bar{x}_{2n-1})\mathcal{B}_{n,n+1}, \quad (20) \end{aligned}$$

with

$$\mathcal{B}_{n,m} = \frac{\prod_{1\le i<j\le n}(\bar{x}_{2i-1}^2 - \bar{x}_{2j-1}^2)\prod_{1\le i<j\le m}(x_{2i}^2 - x_{2j}^2)}{\prod_{1\le i<j\le n+m}(u_i + u_j)}\,. \tag{21}$$

Associated with the particles and anti-particles we introduced here the quantities $x_i = \exp(\theta_i)$ and $\bar{x}_i = \exp(\bar{\theta}_i)$, respectively. The variable u_i can be either of them. We also employed the elementary symmetric polynomials $\sigma_k(x_1, \ldots, x_n)$. The remaining form factors are zero due to the U(1)-symmetry of the Lagrangian. One may easily verify that the expressions (19) and (20) indeed satisfy the consistency equations (4)-(7) with $\gamma_{\bar{\alpha}}^{\mu\alpha} = -1$ and $\gamma_{\bar{\alpha}}^{\sigma\alpha} = 1$ for $\alpha = 1, 2$. We also compute[1] the form factors associated to the trace of the energy-momentum tensor

$$F_2^{T^\mu{}_\mu|\bar{\alpha}\alpha}(\theta, \tilde{\theta}) = F_2^{T^\mu{}_\mu|\alpha\bar{\alpha}}(\theta, \tilde{\theta}) = -2\pi i m_\alpha^2 \sinh\frac{\theta - \tilde{\theta}}{2}\,, \tag{22}$$

which plays a distinct role in the ultraviolet limit.

4. The Federbush Model

The Federbush model[13] was proposed forty years ago as a prototype for an exactly solvable quantum field theory which obeys the Wightman axioms.[14] It contains two different massive particles Ψ_1 and Ψ_2. A special feature of this model is that the related vector currents $J_\alpha^\mu = \bar{\Psi}_\alpha \gamma^\mu \Psi_\alpha$, $\alpha \in \{1, 2\}$, whose analogues occur squared in the massive Thirring model, enter the Lagrangian density of the Federbush model in a parity breaking manner

$$\mathcal{L}_{\mathrm{F}} = \sum\nolimits_{\alpha=1,2} \bar{\Psi}_\alpha(i\gamma^\mu\partial_\mu - m_\alpha)\Psi_\alpha - 2\pi\lambda\varepsilon_{\mu\nu}J_1^\mu J_2^\nu \tag{23}$$

due to the presence of the Levi-Civita pseudotensor ε. The scattering matrix was found to be[14,16]

$$S^{\mathrm{FB}} = -\begin{pmatrix} 1 & 1 & e^{-2\pi i\lambda} & e^{2\pi i\lambda} \\ 1 & 1 & e^{2\pi i\lambda} & e^{-2\pi i\lambda} \\ e^{2\pi i\lambda} & e^{-2\pi i\lambda} & 1 & 1 \\ e^{-2\pi i\lambda} & e^{2\pi i\lambda} & 1 & 1 \end{pmatrix}. \tag{24}$$

For the rows and columns we adopt here the ordering $\{1, \bar{1}, 2, \bar{2}\}$. In close relation to the free fermionic theory one may also introduce the analogue fields to the disorder and order fields in the Federbush model

$$\Phi_\alpha^\lambda(x) = :\exp[\Omega_\alpha^\lambda(x)]: = \vdots\exp[-2\sqrt{\pi}i\lambda\phi_\alpha(x)]\vdots \tag{25}$$

$$\Sigma_\alpha^\lambda(x) = :\int \frac{dp_\alpha^1}{2\pi p_\alpha^0}(a_\alpha(p)e^{-ip_\alpha\cdot x} + a_{\bar{\alpha}}^\dagger(p)e^{ip_\alpha\cdot x})\,\Phi_\alpha^\lambda(x):, \tag{26}$$

where the κ-function related to Ω is

$$\hat{\kappa}^1(\theta, \theta') = -\hat{\kappa}^2(-\theta, -\theta') = \frac{i\sin(\pi\lambda)e^{-\lambda(\theta-\theta')}}{2\cosh\frac{1}{2}(\theta - \theta')}. \tag{27}$$

The last equality in (25) was found by Lehmann and Stehr,[15] who showed the remarkable fact that the operator $\Phi_\alpha^\lambda(x)$ can be viewed in two equivalent ways. On one hand it can be defined through triple ordered free Bosons $\phi_\alpha(x)$, defined as $:e^{\kappa\phi}: = e^{\kappa\phi}/\left\langle e^{\kappa\phi}\right\rangle$ for κ being some constant, and on the other hand by means of a conventional fermionic Wick ordered expression. We compute[1] the following equal time exchange relations for $\alpha,\beta = 1,2$

$$\psi_\alpha(x)\Phi_\beta^\lambda(y) = \Phi_\beta^\lambda(y)\psi_\alpha(x)\, e^{2\pi i(-1)^\beta\lambda\delta_{\alpha\beta}\Theta(x^1-y^1)}, \tag{28}$$

$$-\psi_\alpha(x)\Sigma_\beta^\lambda(y) = \Sigma_\beta^\lambda(y)\psi_\alpha(x)\, e^{2\pi i(-1)^\beta\lambda\delta_{\alpha\beta}\Theta(x^1-y^1)}, \tag{29}$$

$$\Phi_\alpha^\lambda(x)\Phi_\beta^\lambda(y) = \Phi_\beta^\lambda(y)\Phi_\alpha^\lambda(x) \tag{30}$$

$$\Sigma_\alpha^\lambda(x)\Sigma_\beta^\lambda(y) = \Sigma_\beta^\lambda(y)\Sigma_\alpha^\lambda(x)\, e^{2\pi i(-1)^\beta\lambda\delta_{\alpha\beta}}\,. \tag{31}$$

where $\Theta(x)$ is the Heaviside step function. With the relevant exchange relations at our disposal, we can, according to (8), read off the factors of local commutativity for the operators under consideration

$$\gamma_\alpha^{\Phi_\beta^\lambda} = -\gamma_\alpha^{\Sigma_\beta^\lambda} = e^{2\pi i(-1)^\beta\lambda\delta_{\alpha\beta}} \quad\text{and}\quad \gamma_{\bar\alpha}^{\Phi_\beta^\lambda} = -\gamma_{\bar\alpha}^{\Sigma_\beta^\lambda} = e^{-2\pi i(-1)^\beta\lambda\delta_{\alpha\beta}}\,. \tag{32}$$

Proceeding again in the same way as in the previous section, we obtain as closed expressions for the n-particle form factors

$$\begin{aligned} &F_{2n}^{\Phi_1^\lambda|n\times\bar 1 1}(\bar x_1, x_2 \ldots \bar x_{2n-1}, x_{2n}) = (-1)^n F_{2n}^{\Phi_2^{-\lambda}|n\times\bar 2 2}(\bar x_1, x_2 \ldots \bar x_{2n-1}, x_{2n}) = \\ &F_{2n}^{\Phi_{\bar 1}^{-\lambda}|n\times\bar 1 1}(\bar x_1, x_2 \ldots \bar x_{2n-1}, x_{2n}) = (-1)^n F_{2n}^{\Phi_{\bar 2}^{\lambda}|n\times\bar 2 2}(\bar x_1, x_2 \ldots \bar x_{2n-1}, x_{2n}) = \\ &\qquad i^n 2^{n-1}\sin^n(\pi\lambda)\sigma_n(\bar x_1 \ldots \bar x_{2n-1})^{\lambda+\frac{1}{2}}\sigma_n(x_2 \ldots x_{2n})^{\frac{1}{2}-\lambda}\mathcal{B}_{n,n}, \end{aligned} \tag{33}$$

$$\begin{aligned} &\tilde F_{2n+1}^{\Sigma_1^\lambda|1(n\times\bar 1 1)}(\theta_1,\ldots,\theta_{2n+1}) = (-1)^n \tilde F_{2n+1}^{\Sigma_2^{-\lambda}|2(n\times\bar 2 2)}(\theta_1,\ldots,\theta_{2n+1}) = \\ &\tilde F_{2n+1}^{\Sigma_{\bar 1}^{-\lambda}|1(n\times\bar 1 1)}(\theta_1,\ldots,\theta_{2n+1}) = (-1)^n \tilde F_{2n+1}^{\Sigma_{\bar 2}^{\lambda}|2(n\times\bar 2 2)}(\theta_1,\ldots,\theta_{2n+1}) = \frac{\sin^n(\pi\lambda)}{2} \\ &\frac{(2i)^n\sigma_n(\bar x_2 \ldots \bar x_{2n})^{\lambda+\frac{1}{2}}}{\sigma_n(x_1 \ldots x_{2n+1})^{\lambda-\frac{1}{2}}}\prod_{1\le i<j\le n}(\bar x_{2i}-\bar x_{2j})\sum_k \frac{i^{k+1}\prod\limits_{j<l;j,l\neq k}(x_j-x_l)}{(x_k)^{\frac{1}{2}-\lambda}\prod\limits_{j\neq k}\prod\limits_{l}(x_j+\bar x_l)}. \end{aligned} \tag{34}$$

We may now convince ourselves, that the expressions for $F_{2n}^{\Phi_\alpha^\lambda|n\times\bar\alpha\alpha}$ indeed satisfy the consistency equations (4)-(7). However, the expressions of $\tilde F_{2n+1}^{\Sigma_\alpha^\lambda|\alpha(n\times\bar\alpha\alpha)}$ only satisfy the consistency equations (4)-(7) for $\lambda = 1/2$. This reflects the very important fact that $\Sigma_\alpha^\lambda(x)$ is only a mutually local operator for this value of λ, see equation (31), unlike $\Phi_\alpha^\lambda(x)$ which is mutually local for all value of λ. Thus, the equations (4)-(7) select out solutions corresponding to operators which are mutually local.

The form factors related to the trace of the energy-momentum tensor turn out to be the same as the ones for the complex free Fermion.

5. Momentum space cluster properties

An interesting operator related property which the form factors satisfy is the momentum space cluster decomposition

$$\lim_{\Delta\to\infty} F^{\mathcal{O}}_{k+l}(\theta_1\dots\theta_k,\theta_{k+1}+\Delta\dots\theta_{k+l}+\Delta) = F^{\mathcal{O}'}_k(\theta_1\dots\theta_k)F^{\mathcal{O}''}_l(\theta_{k+1}\dots\theta_{k+l}) \ , \quad (35)$$

Writing instead of the matrix elements only the operators, we obtained[1] formally the following decomposition

$$\Phi^\lambda_\alpha \longrightarrow \Phi^\lambda_\alpha \times \Phi^\lambda_\alpha \qquad \sigma_\alpha \longrightarrow \begin{cases} \mu_\alpha \times \sigma_\alpha \\ \mu_{\bar\alpha} \times \sigma_\alpha \end{cases} \qquad \mu_\alpha \longrightarrow \begin{cases} \mu_\alpha \times \mu_\alpha \\ \sigma_\alpha \times \sigma_{\bar\alpha} \end{cases} \quad (36)$$

together with the equations for $\alpha \leftrightarrows \bar\alpha$. This means the stated operator content closes consistently under the action of the cluster decomposition operators. We also observe that non-selfclustering, i.e. $\mathcal{O} \neq \mathcal{O}' \neq \mathcal{O}''$, is possible. Unlike the self-clustering, which can be explained for the bosonic case with the help of Weinberg's power counting argument, this property is not yet understood from general principles.

6. Lie algebraically coupled Federbush models

The Federbush model as investigated in the previous section only contains two types of particles. In this section we propose a new Lagrangian, which admits a much larger particle content. The theories are not yet as complex as the homogeneous sine-Gordon (HSG) models, but they can also be obtained from them in a certain limit such that they will always constitute a benchmark for these class of theories.

Let us consider $\ell \times \tilde\ell$-real (Majorana) free Fermions $\psi_{a,j}(x)$, now labeled by two quantum numbers $1 \le a \le \ell$, $1 \le j \le \tilde\ell$ and described by the Dirac Lagrangian density $\mathcal{L}_{\rm FF}$. We perturb this system with a bilinear term in the vector currents $J^\mu_{a,j} = \bar\Psi_{a,j}\gamma^\mu\Psi_{a,j}$

$$\mathcal{L}_{\rm CF} = \sum_{a=1}^{\ell}\sum_{j=1}^{\tilde\ell} \bar\Psi_{a,j}(i\gamma^\mu\partial_\mu - m_{a,j})\Psi_{a,j} - \frac{1}{2}\pi\varepsilon_{\mu\nu}\sum_{a,b=1}^{\ell}\sum_{j,k=1}^{\tilde\ell} J^\mu_{a,j}J^\nu_{b,k}\Lambda^{jk}_{ab} \ , \quad (37)$$

and denote the new fields in $\mathcal{L}_{\rm CF}$ by $\Psi_{a,j}$. Furthermore, we introduced $\ell^2 \times \tilde\ell^2$ dimensional coupling constant dependent matrix Λ^{jk}_{ab}, whose further properties we leave unspecified at this stage. We computed[1] the related S-matrix to

$$S^{jk}_{ab} = -e^{i\pi\Lambda^{jk}_{ab}} \ . \quad (38)$$

where due to the crossing and unitarity relations we have the constraints

$$\Lambda^{jk}_{ab} = -\Lambda^{kj}_{ba} + 2\mathbb{Z} \qquad \text{and} \qquad \Lambda^{jk}_{ab} = \Lambda^{\bar k j}_{\bar b a} + 2\mathbb{Z} \quad (39)$$

on the constants Λ. Taking $\Lambda^{jk}_{ab} = 2\lambda_{ab}\varepsilon_{jk}\tilde I_{jk}K^{-1}_{a\bar b}$, with K, I being the Cartan and incidence matrix, respectively, provides the limit of the HSG-models.

7. The ultraviolet limit

The ultraviolet Virasoro central charge of the theory itself can be computed from the knowledge of the form factors of the trace of the energy-momentum tensor[17] by means of the expansion

$$c_{\rm uv} = \sum_{n=1}^{\infty} \sum_{\mu_1 \dots \mu_n} \frac{9}{n!(2\pi)^n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{d\theta_1 \dots d\theta_n \left| F_n^{T^\mu{}_\mu | \mu_1 \dots \mu_n}(\theta_1, \dots, \theta_n) \right|^2}{\left(\sum_{i=1}^n m_{\mu_i} \cosh \theta_i \right)^4} . \quad (40)$$

In a similar way one may compute the scaling dimension of the operator $\mathcal{O}$ from the knowledge of its n-particle form factors[18]

$$\Delta^{\mathcal{O}}_{\rm uv} = -\frac{1}{2\langle \mathcal{O} \rangle} \sum_{n=1}^{\infty} \sum_{\mu_1 \dots \mu_n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{d\theta_1 \dots d\theta_n}{n!(2\pi)^n \left(\sum_{i=1}^n m_{\mu_i} \cosh \theta_i \right)^2}$$
$$\times F_n^{T^\mu{}_\mu | \mu_1 \dots \mu_n}(\theta_1, \dots, \theta_n) \left(F_n^{\mathcal{O} | \mu_1 \dots \mu_n}(\theta_1, \dots, \theta_n) \right)^* . \quad (41)$$

In general the expressions (40) and (41) yield the difference between the corresponding infrared and ultraviolet values, but we assumed here already that the theory is purely massive such that the infrared contribution vanishes. Evaluating these formulae, we obtain

$$c_{\rm uv} = 2 \qquad \text{and} \qquad \Delta^{\mu_\alpha}_{\rm uv} = \Delta^{\mu_{\bar{\alpha}}}_{\rm uv} = \frac{1}{16} . \quad (42)$$

for the complex free Fermion and

$$c_{\rm uv} = 2 \qquad \text{and} \qquad \Delta^{\Phi^\lambda_\alpha}_{\rm uv} = \Delta^{\Phi^\lambda_{\bar{\alpha}}}_{\rm uv} = \frac{\lambda^2}{4} . \quad (43)$$

for the Federbush model Note, that $\Delta^{\Phi^{1/2}_\alpha}_{\rm uv} = \Delta^{\Phi^{1/2}_{\bar{\alpha}}}_{\rm uv} = 1/16$, which is the limit to the complex free Fermion. Yet more support for the relation between the $SU(3)_3$-HSG model and the Federbush model comes from the analysis of $\lambda = 2/3$, for which the $SU(3)_3$-HSG S-matrix is related to the one of the Federbush model. In that case we obtain from (43) the values $\Delta^{\Phi^{2/3}_\alpha}_{\rm uv} = \Delta^{\Phi^{2/3}_{\bar{\alpha}}}_{\rm uv} = 1/9$, which is a conformal dimension occurring in the $SU(3)_3$-HSG model. Thus precisely at the value of the coupling constant of the Federbush model at which the $SU(3)_3$-HSG S-matrix reduces to the $S^{\rm FB}$, the operator content of the two models overlaps.

8. Conclusions

We summarize our main results:

We computed explicitly closed formulae for the n-particle form factors of the complex free Fermion and the Federbush model related to various operators.

We carried out this computations in two alternative ways: On the one hand, we represent explicitly the field content (12) as well as the particle creation operators (9) in terms of fermionic Fock operators (10) and computed thereafter directly the

corresponding matrix elements. On the other hand we verified that these expressions satisfy the form factor consistency equations only when the operators under consideration are mutually local, i.e. satisfying (1). It is crucial that the consistency equations contain the factor of local commutativity $\gamma_\mu^{\mathcal{O}}$ as defined in (8). Our analysis strongly suggest that *the form factor consistency equations select out operators, which are mutually local in the sense of (1).*

Our solutions turned out to decompose consistently under the momentum space cluster property. This computations constitute next to the ones in Refs. 19, 20 the first concrete examples of non-selfclustering, i.e. $\mathcal{O} \to \mathcal{O}' \times \mathcal{O}''$ in the sense of (36).

Further support for the identification of the solutions of (4)-(7) with a specific operator was given by an analysis of the ultraviolet limit.

We demonstrated how the scattering matrix of the Federbush model can be obtained as a limit of the $SU(3)_3$-HSG scattering matrix. This "correspondence" also holds for the central charge, which equals 2 in both cases, and the scaling dimension of the disorder operator at a certain value of the coupling constant.

We proposed a Lie algebraic generalization of the Federbush models, by suggesting a new type of Lagrangian. We evaluate from first principles the related scattering matrices, which can also be obtained in a certain limit from the HSG-models.

We expect that the construction of form factors by means of free fermionic Fock fields can be extended to other models by characterizing further the function κ.

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DEFORMATION QUANTIZATION: IS C_1 NECESSARILY SKEW?

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Deformation quantization (of a commutative algebra) is based on the introduction of a new associative product, expressed as a formal series, $f * g = fg + \sum_{n=1}^{\infty} \hbar^n C_n(f,g)$. In the case of the algebra of functions on a symplectic space the first term in the perturbation is often identified with the antisymmetric Poisson bracket. There is a widespread belief that every associative $*$-product is equivalent to one for which $C_1(f,g)$ is antisymmetric and that, in particular, every abelian deformation is trivial. This paper shows that this is far from being the case and illustrates the existence of abelian deformations by physical examples.

1. Introduction

First, a very brief review of quantization:

- H. Weyl looked at quantization as a one-to-one correspondence between functions on phase space on the one hand, and operators in a Hilbert space on the other:[1]

$$\text{function on phasespace } = f \mapsto W(f) = \text{ operator.} \tag{1}$$

- Taking over Weyl's correspondence, Moyal[2] proposed an autonomous formulation of quantum mechanics, in terms of functions on phase space endowed with a new, non-commutative product, called a $*$-product, namely

$$f * g = W^{-1}(W(f)W(g)). \tag{2}$$

- Moyal's idea was incorporated into the new, deformation theory approach to quantization,[3] where one studies more general associative $*$-products viewed as formal series in a parameter

$$f * g = \sum_{n=0}^{\infty} \hbar^n C_n(f,g), \quad C_0(f,g) = fg, \tag{3}$$

and usually

$$C_1(f,g) = \frac{1}{2}\{f,g\}. \tag{4}$$

The bracket $\{,\}$ is the Poisson bracket. Such deformations "in the direction of the Poisson bracket" play a basic role in Drinfel'd's approach to quantum groups.[4] The existence and classification of $*$-products on an arbitrary symplectic manifold were placed in a nice geometric context by the approach of Fedosov;[5] see also Ref. 6.

- Recently, M. Kontsevich has shown[7] that $*$-products exist on arbitrary Poisson manifolds. See also Ref. 8. This is a significant achievement that is currently exciting much interest among mathematicians.

This summary suggests that the first order term, $C_1(f,g)$, is always antisymmetric in its two arguments. But there is a footnote to the story, namely:

- Geometric, Souriau-Kostant quantization on co-adjoint orbits,[9,10] and the generalization of this method within the $*$-product-deformation approach (see for example Ref. 11) is a parallel development. As we shall see, the most interesting case, that of quantization on a singular orbit, is characterized by a $*$-product for which $C_1(f,g)$ is not skew.

We also cite the program of quantization of Nambu mechanics that ran aground on the belief that every abelian $*$-product is trivial.[12] This is mistaken, and for two independent reasons; first, because it is false on varieties with singularities and second, because it holds only to first order in the deformation parameter.

2. Quantization on algebraic varieties

On the manifold $\mathbb{R}$, with global coordinates $x_1, \cdots, x_n$, consider the algebra

$$A = \mathbb{C}[x_1, \cdots, x_n] \tag{5}$$

of polynomials in n variables. On this algebra, consider an associative deformation of the ordinary product (the commutative product of functions) in the form

$$f * g = \sum_{n=0}^{\infty} \hbar^n C_n(f,g), \quad C_0(f,g) = fg. \tag{6}$$

Associativity, to first order in $\hbar$, is the statement that $dC_1 = 0$, where d is the Hochschild differential,

$$dC(f,g,h) = fC(g,h) - C(fg,h) + C(f,gh) - C(f,g)h. \tag{7}$$

Triviality of the $*$-product (see below) is also expressed in terms of the Hochschild differential: The above $*$-product is trivial to first order if there is a one-cochain E_1 such that $C_1 = dE_1$, where

$$dE(f,g) := fE(g) - E(fg) + E(f)g. \tag{8}$$

The following theorem tells us that, to first order in $\hbar$, all abelian $*$-products on $\mathbb{R}^n$, and indeed all abelian $*$-products on any smooth manifold, are trivial.

Theorem: (Hochschild, Kostant, Rosenberg[13]) If A is the generated algebra of functions on a smooth manifold then the space $H^n(A)$ is the space of alternating n-forms.

This implies that, to first order in $\hbar$, every $*$-product on a smooth manifold is equivalent to one for which C_1 is antisymmetric.

3. Escape route number one

Let us consider algebraic varieties with singularities, for example the following.

$$M = \mathbb{R}/(x^2 - y^2), \quad A = \mathbb{C}[x, y]/(x^2 - y^2). \tag{9}$$

Expand $f \in A$, $\quad f(x, y) = f_1(x) + y f_2(x)$ and define a deformed product by the formula

$$f * g = fg + \hbar f_2 g_2. \tag{10}$$

It is associative to all orders, and not trivial. Proof of non-triviality:

$$A_\hbar = \mathbb{C}[x, y]/(x^2 - y^2 + \hbar). \tag{11}$$

If $\hbar$ is real and not zero, then this is the coordinate algebra on a smooth manifold, clearly not isomorphic to the coordinate algebra of the cone $x^2 = y^2$.

A large class of examples is provided by algebraic varieties,

$$M = \mathbb{R}^n/R, \quad R = \text{a set of polynomial relations}, \tag{12}$$

$$A = \mathbb{C}[x_1, \cdots, x_n]/R. \tag{13}$$

The calculation of cohomology is often fairly simple and rests on the fact that the Hochschild cohomology of A is equivalent to its restriction to linear, closed chains. (See Ref. 14.)

In our example, the closed, linear chains are

$$x \wedge y, \quad \text{and} \quad x \otimes x - y \otimes y, \tag{14}$$

$$d(x \wedge y) = xy - yx = 0, \quad d(x \otimes x - y \otimes y) = x^2 - y^2 = 0. \tag{15}$$

The 2-cochain C is closed if it is symmetric and it is exact if $C(x \otimes x - y \otimes y) \in A$. A representative of the unique non-trivial equivalence class of 2-forms is

$$C(x \wedge y) = 0, \quad C(x \otimes x - y \otimes y) = k \in \mathbb{C} - \{0\}. \tag{16}$$

Besides Nambu mechanics, this has applications to the problem of quantization on coadjoint orbits.

Quantization on coadjoint orbits, an example

The algebra is the universal enveloping algebra of $sl(2,R)$, basis x_1, x_2, x_0. The unique singular orbit is $Q := x_1^2 + x_2^2 - x_0^2 = 0$. Invariant quantization is obtained via the Weyl-type correspondence:

$$x_i * x_j - x_j * x_i = \hbar \epsilon_{ijk} x_k, \tag{17}$$

$$P_n^*(a) = P_n(a), \quad a = \sum_{i=1,2,0} a^i x_i, \tag{18}$$

where P_n are Legendre polynomials and P_n^* are the same symmetric $*$-polynomials, and the assignment of a value to the Casimir operator

$$Q^* : x_1 * x_1 + x_2 * x_2 - x_0 * x_0 \mapsto \hbar l(l+1). \tag{19}$$

For $n = 2$, on the singular orbit $Q = 0$,

$$\frac{1}{2}(x_i * x_j + i, j) - \frac{1}{3}\delta_{ij} Q^* = x_i x_j, \tag{20}$$

and thus

$$x_i * x_j = x_i x_j + \hbar\left(\frac{1}{2}\epsilon_{ijk} x_k + \frac{1}{3}\delta_{ij} l(l+1)\right). \tag{21}$$

This makes use of the cohomologically nontrivial symmmetric 2-cochain $C_2(x_i, x_j) \propto \delta_{ij}$ as well as the more familiar antisymmetric cochain.

4. Escape route number two

In the examples given, associativity is satisfied to all orders with $C_n = 0, \quad n > 1$. Non-triviality is verified to lowest order and of course cannot be changed in higher orders.

In the example that follows we shall again have associativity to all orders, and triviality in the lowest order, but this does not imply triviality in higher orders.

Triviality of a $*$-product is the statement that there is a map $E : A \to A$, of the form

$$E(f) = f + \sum_{n=1}^{\infty} \hbar^n E_n(f), \tag{22}$$

such that

$$f * g = E^{-1}(E(f)E(g)). \tag{23}$$

To first order in $\hbar$ this says that

$$C_1(f,g) = E_1(f)g - E_1(fg) + fE_1(g) =: dE_1(f,g). \tag{24}$$

Here is an example of an abelian $*$-product that is trivial to first order but nontrivial when taken to all orders.

Let $M =$ Minkowsky space, $A = \mathbb{C}[x_1, \cdots, x_4]$. Decompose $f \in A$ into even and odd parts.: $f = (f_+, f_-)$, and define:

$$f * g = fg - \rho x^2 f_- g_- = \left(f_+ g_+ + f_- g_- (1 - \rho x^2), \quad f_+ g_- + f_- g_+ \right) \tag{25}$$

Now, let M' =3+2-dimensional anti-De Sitter space, more precisely the cone in $\mathbb{R}^5$:

$$M' = \mathbb{R}^5 / (\rho x^2 + y^2 - 1), \tag{26}$$

and

$$A' = \mathbb{C}[x_1, \cdots, x_4, y]^{\mathrm{e}} / (\rho x^2 + y^2 - 1), \tag{27}$$

where $[...]^e$ means polynomials of even order. Decompose $f \in A'$ as follows, $f = f_+(x) + y f_-(x)$, then

$$fg = f_+ g_+ + f_- g_- (1 - \rho x^2) + y(f_+ g_- + f_- g_+). \tag{28}$$

Therefore, the deformed, $*$-product algebra of functions on Minkowski space is isomorphic to the ordinary algebra of even functions on AdS. But A and A' are not isomorphic and the $*$-product is therefore not trivial. (For more details see Ref. 14.) It seems possible that this provides a new approach to physics in AdS that could help to overcome some of the problems of interpretation.

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GENERALIZED GINSPARG–WILSON ALGEBRA AND INDEX THEOREM ON THE LATTICE

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Recent studies of the topological properties of a general class of lattice Dirac operators are reported. This is based on a specific algebraic realization of the Ginsparg-Wilson relation in the form $\gamma_5(\gamma_5 D)+(\gamma_5 D)\gamma_5 = 2a^{2k+1}(\gamma_5 D)^{2k+2}$ where k stands for a non-negative integer. The choice $k=0$ corresponds to the commonly discussed Ginsparg-Wilson relation and thus to the overlap operator. It is shown that local chiral anomaly and the instanton-related index of all these operators are identical. The locality of all these Dirac operators for vanishing gauge fields is proved on the basis of explicit construction, but the locality with dynamical gauge fields has not been fully established yet.

1. Introduction

Recent developments in the treatment of fermions in lattice gauge theory are based on a hermitian lattice Dirac operator $\gamma_5 D$ which satisfies the Ginsparg-Wilson relation[1]

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D \tag{1}$$

where the lattice spacing a is utilized to make a dimensional consideration transparent, and γ_5 is a hermitian chiral Dirac matrix. An explicit example of the operator satisfying (1.1) and free of species doubling has been given by Neuberger.[2] The relation (1.1) led to an interesting analysis of the notion of index in lattice gauge theory.[3] This index theorem in turn led to a new form of chiral symmetry, and the chiral anomaly is obtained as a non-trivial Jacobian factor under this modified chiral transformation.[4] This chiral Jacobian is regarded as a lattice generalization of the continuum path integral.[5] The very detailed analyses of the lattice chiral Jacobian have been performed.[6] It is also possible to formulate the lattice index theorem in a manner analogous to the continuum index theorem.[7–9] An interesting chirality sum rule, which relates the number of zero modes to that of the heaviest states, has also been noticed.[10] See Refs. 11 for reviews of these developments.

We have recently discussed the possible generalization of (1.1) and its implications.[12] To be specific, we have discussed a generalization of the algebra (1.1) in

the form

$$\gamma_5(\gamma_5 D) + (\gamma_5 D)\gamma_5 = 2a^{2k+1}(\gamma_5 D)^{2k+2} \tag{2}$$

where k stands for a non-negative integer and $k = 0$ corresponds to the ordinary Ginsparg-Wilson relation. When one defines

$$H \equiv \gamma_5 a D \tag{3}$$

(1.2) is rewritten as

$$\gamma_5 H + H\gamma_5 = 2H^{2k+2} \tag{4}$$

or equivalently

$$\Gamma_5 H + H\Gamma_5 = 0 \tag{5}$$

where we defined

$$\Gamma_5 \equiv \gamma_5 - H^{2k+1}. \tag{6}$$

Note that both of H and Γ_5 are hermitian operators.

It has been shown that all the good topological properties of the overlap operator[2] is retained in this generalization.[12,13] The practical applications of this generalization are not known at this moment. We however mention the characteristic properties of this generalization: The spectrum near the continuum configuration is closer to that of continuum theory and the chiral symmetry breaking terms become more irrelevant in the continuum limit for $k \geq 1$. The operator however spreads over more lattice points for large k.

2. Representation of the general algebra

We first discuss a general representation of the algebraic relation (1.5). The relation (1.5) suggests that if

$$H\phi_n = a\lambda_n \phi_n, \quad (\phi_n, \phi_n) = 1 \tag{7}$$

with a real eigenvalue $a\lambda_n$ for the hermitian operator H, then

$$H(\Gamma_5 \phi_n) = -a\lambda_n(\Gamma_5 \phi_n). \tag{8}$$

Namely, the eigenvalues λ_n and $-\lambda_n$ are always paired if $\lambda_n \neq 0$ and $(\Gamma_5\phi_n, \Gamma_5\phi_n) \neq 0$. We also note the relation, which is derived by sandwiching the relation (1.4) by ϕ_n,

$$(\phi_n, \gamma_5\phi_n) = (a\lambda_n)^{2k+1} \quad for \quad \lambda_n \neq 0. \tag{9}$$

Consequently

$$|(a\lambda_n)^{2k+1}| = |(\phi_n, \gamma_5\phi_n)| \leq ||\phi_n||||\gamma_5\phi_n|| = 1. \tag{10}$$

Namely, all the possible eigenvalues are bounded by

$$|\lambda_n| \leq \frac{1}{a}. \tag{11}$$

We thus evaluate the norm of $\Gamma_5\phi_n$

$$\begin{aligned}(\Gamma_5\phi_n, \Gamma_5\phi_n) &= (\phi_n, (\gamma_5 - H^{2k+1})(\gamma_5 - H^{2k+1})\phi_n) \\ &= (\phi_n, (1 - H^{2k+1}\gamma_5 - \gamma_5 H^{2k+1} + H^{2(2k+1)})\phi_n) \\ &= [1 - (a\lambda_n)^{2(2k+1)}] \\ &= [1 - (a\lambda_n)^2][1 + (a\lambda_n)^2 + ... + (a\lambda_n)^{4k}] \end{aligned} \tag{12}$$

where we used (2.3). By remembering that all the eigenvalues are real, we find that ϕ_n is a "highest" state

$$\Gamma_5\phi_n = 0 \tag{13}$$

only if

$$[1 - (a\lambda_n)^2] = (1 - a\lambda_n)(1 + a\lambda_n) = 0 \tag{14}$$

for the Euclidean positive definite inner product $(\phi_n, \phi_n) \equiv \sum_x \phi_n^\dagger(x)\phi_n(x)$.
We thus conclude that the states ϕ_n with $\lambda_n = \pm\frac{1}{a}$ are *not* paired by the operation $\Gamma_5\phi_n$ and

$$\gamma_5 D\phi_n = \pm\frac{1}{a}\phi_n, \quad \gamma_5\phi_n = \pm\phi_n \tag{15}$$

respectively. These eigenvalues are in fact the maximum or minimum of the possible eigenvalues of H/a due to (2.5).

As for the vanishing eigenvalues $H\phi_n = 0$, we find from (1.4) that $H\gamma_5\phi_n = 0$, namely, $H[(1 \pm \gamma_5)/2]\phi_n = 0$. We can thus choose

$$\gamma_5 D\phi_n = 0, \quad \gamma_5\phi_n = \phi_n \quad or \quad \gamma_5\phi_n = -\phi_n. \tag{16}$$

To summarize the analyses so far, all the normalizable eigenstates ϕ_n of $\gamma_5 D = H/a$ are categorized into the following 3 classes:
(i) $n_\pm$ ("zero modes"),

$$\gamma_5 D\phi_n = 0, \quad \gamma_5\phi_n = \pm\phi_n, \tag{17}$$

(ii) $N_\pm$ ("highest states"),

$$\gamma_5 D\phi_n = \pm\frac{1}{a}\phi_n, \quad \gamma_5\phi_n = \pm\phi_n, \quad respectively, \tag{18}$$

(iii) "paired states" with $0 < |\lambda_n| < 1/a$,

$$\gamma_5 D\phi_n = \lambda_n\phi_n, \quad \gamma_5 D(\Gamma_5\phi_n) = -\lambda_n(\Gamma_5\phi_n). \tag{19}$$

Note that $\Gamma_5(\Gamma_5\phi_n) \propto \phi_n$ for $0 < |\lambda_n| < 1/a$.

We thus obtain the index relation[3,4]

$$\begin{aligned}
\mathrm{Tr}\Gamma_5 &\equiv \sum_n (\phi_n, \Gamma_5 \phi_n) \\
&= \sum_{\lambda_n=0} (\phi_n, \Gamma_5 \phi_n) + \sum_{0<|\lambda_n|<1/a} (\phi_n, \Gamma_5 \phi_n) + \sum_{|\lambda_n|=1/a} (\phi_n, \Gamma_5 \phi_n) \\
&= \sum_{\lambda_n=0} (\phi_n, \Gamma_5 \phi_n) \\
&= \sum_{\lambda_n=0} (\phi_n, (\gamma_5 - H^{2k+1}) \phi_n) \\
&= \sum_{\lambda_n=0} (\phi_n, \gamma_5 \phi_n) \\
&= n_+ - n_- = index
\end{aligned} \tag{20}$$

where $n_\pm$ stand for the number of normalizable zero modes with $\gamma_5 \phi_n = \pm \phi_n$ in the classification (i) above. We here used the fact that $\Gamma_5 \phi_n = 0$ for the "highest states" and that ϕ_n and $\Gamma_5 \phi_n$ are orthogonal to each other for $0 < |\lambda_n| < 1/a$ since they have eigenvalues with opposite signatures.

On the other hand, the relation $\mathrm{Tr}\gamma_5 = 0$, which is expected to be valid in (finite) lattice theory, leads to (by using (2.3))

$$\begin{aligned}
\mathrm{Tr}\gamma_5 &= \sum_n (\phi_n, \gamma_5 \phi_n) \\
&= \sum_{\lambda_n=0} (\phi_n, \gamma_5 \phi_n) + \sum_{\lambda_n \neq 0} (\phi_n, \gamma_5 \phi_n) \\
&= n_+ - n_- + \sum_{\lambda_n \neq 0} (a\lambda_n)^{2k+1} = 0.
\end{aligned} \tag{21}$$

In the last line of this relation, all the states except for the "highest states" with $\lambda_n = \pm 1/a$ cancel pairwise for $\lambda_n \neq 0$. We thus obtain a chirality sum rule[10]

$$n_+ + N_+ = n_- + N_- \tag{22}$$

where $N_\pm$ stand for the number of "highest states" with $\gamma_5 \phi_n = \pm \phi_n$ in the classification (ii) above. These relations show that the chirality asymmetry at vanishing eigenvalues is balanced by the chirality asymmetry at the largest eigenvalues with $|\lambda_n| = 1/a$. It was argued in Ref. 4 that $N_\pm$ states are the topological (instanton-related) excitations of the would-be species doublers.

We have thus established that the representation of all the algebraic relations (1.2) has a similar structure. In the next section, we show that the index $n_+ - n_-$ is identical to all these algebraic relations if the operator $\gamma_5 D$ satisfies suitable conditions.

3. Chiral Jacobian and the index relation

The Euclidean path integral for a fermion is defined by

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp[\int \bar{\psi}D\psi] \tag{23}$$

where

$$\int \bar{\psi}D\psi \equiv \sum_{x,y} \bar{\psi}(x)D(x,y)\psi(y) \tag{24}$$

and the summation runs over all the points on the lattice. The relation (1.5) is re-written as

$$\gamma_5\Gamma_5\gamma_5 D + D\Gamma_5 = 0 \tag{25}$$

and thus the Euclidean action is invariant under the global "chiral" transformation[4]

$$\begin{aligned}\bar{\psi}(x) \to \bar{\psi}'(x) &= \bar{\psi}(x) + i\sum_z \bar{\psi}(z)\epsilon\gamma_5\Gamma_5(z,x)\gamma_5 \\ \psi(y) \to \psi'(y) &= \psi(y) + i\sum_w \epsilon\Gamma_5(y,w)\psi(w)\end{aligned} \tag{26}$$

with an infinitesimal constant parameter ϵ. Under this transformation, one obtains a Jacobian factor

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = J\mathcal{D}\bar{\psi}\mathcal{D}\psi \tag{27}$$

with

$$J = \exp[-2i\mathrm{Tr}\epsilon\Gamma_5] = \exp[-2i\epsilon(n_+ - n_-)] \tag{28}$$

where we used the index relation (2.14).

We now relate this index appearing in the Jacobian to the Pontryagin index of the gauge field in a smooth continuum limit by following the procedure in Ref. 12. We start with

$$\mathrm{Tr}\{\Gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\} = \mathrm{Tr}\{\Gamma_5 f(\frac{(H/a)^2}{M^2})\} = n_+ - n_- \tag{29}$$

Namely, the index is not modified by any regulator $f(x)$ with $f(0) = 1$ and $f(x)$ rapidly going to zero for $x \to \infty$, as can be confirmed by using (2.14). This means that you can use *any* suitable $f(x)$ in the evaluation of the index by taking advantage of this property.

We then consider a local version of the index

$$\mathrm{tr}\{\Gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\}(x,x) = \mathrm{tr}\{(\gamma_5 - H^{2k+1})f(\frac{(\gamma_5 D)^2}{M^2})\}(x,x) \tag{30}$$

where trace stands for Dirac and Yang-Mills indices; Tr in (3.7) includes a sum over the lattice points x. A local version of the index is not sensitive to the precise boundary condition , and one may take an infinite volume limit of the lattice in the above expression.

We now examine the continuum limit $a \to 0$ of the above local expression (3.8)[a]. We first observe that the term

$$\mathrm{tr}\{H^{2k+1} f(\frac{(\gamma_5 D)^2}{M^2})\} \tag{31}$$

goes to zero in this limit. The large eigenvalues of $H = a\gamma_5 D$ are truncated at the value $\sim aM$ by the regulator $f(x)$ which rapidly goes to zero for large x. In other words, the global index of the operator $\mathrm{Tr} H^{2k+1} f(\frac{(\gamma_5 D)^2}{M^2}) \sim O(aM)^{2k+1} \to 0$ for $a \to 0$ with fixed M.

We thus examine the small a limit of

$$\mathrm{tr}\{\gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\}. \tag{32}$$

The operator appearing in this expression is well regularized by the function $f(x)$, and we evaluate the above trace by using the plane wave basis to extract an explicit gauge field dependence. We consider a square lattice where the momentum is defined in the Brillouin zone

$$-\frac{\pi}{2a} \leq k_\mu < \frac{3\pi}{2a}. \tag{33}$$

We assume that the operator D is free of species doubling, which is proved for the explicit construction of D; in other words, the operator D blows up rapidly ($\sim \frac{1}{a}$) for small a in the momentum region corresponding to species doublers. The contributions of doublers are eliminated by the regulator $f(x)$ in the above expression, since

$$\mathrm{tr}\{\gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\} \sim (\frac{1}{a})^4 f(\frac{1}{(aM)^2}) \to 0 \tag{34}$$

for $a \to 0$ if one chooses $f(x) = e^{-x}$, for example.

We thus examine the above trace in the momentum range of the physical species

$$-\frac{\pi}{2a} \leq k_\mu < \frac{\pi}{2a}. \tag{35}$$

We obtain the limiting $a \to 0$ expression

$$\begin{aligned}
&\lim_{a\to 0} \mathrm{tr}\{\gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\}(x,x) \\
&= \lim_{a\to 0} \mathrm{tr} \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 f(\frac{(\gamma_5 D)^2}{M^2}) e^{ikx} \\
&= \lim_{L\to\infty} \lim_{a\to 0} \mathrm{tr} \int_{-L}^{L} \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 f(\frac{(\gamma_5 D)^2}{M^2}) e^{ikx} \\
&= \lim_{L\to\infty} \mathrm{tr} \int_{-L}^{L} \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 f(\frac{(i\gamma_5 \not{D})^2}{M^2}) e^{ikx} \\
&\equiv \mathrm{tr}\{\gamma_5 f(\frac{\not{D}^2}{M^2})\}
\end{aligned} \tag{36}$$

[a]This continuum limit corresponds to the so-called "naive" continuum limit in the context of lattice gauge theory.

where we first take the limit $a \to 0$ with fixed k_μ in $-L \leq k_\mu \leq L$, and then take the limit $L \to \infty$. This procedure is justified if the integral is well convergent.[12] We also *assumed* that the operator D satisfies the following relation in the limit $a \to 0$

$$\begin{aligned} De^{ikx}h(x) &\to e^{ikx}(-\not{k} + i\,\not{\partial} - g\,\not{A})h(x) \\ &= i(\not{\partial} + ig\,\not{A})(e^{ikx}h(x)) \equiv i\,\not{D}(e^{ikx}h(x)) \end{aligned} \tag{37}$$

for any *fixed* k_μ, $(-\frac{\pi}{2a} < k_\mu < \frac{\pi}{2a})$, and a sufficiently smooth function $h(x)$. The function $h(x)$ corresponds to the gauge potential in our case, which in turn means that the gauge potential $A_\mu(x)$ is assumed to vary very little over the distances of the elementary lattice spacing.

Our final expression (3.14) in the limit $M \to \infty$ reproduces the Pontryagin number in the continuum formulation (with $\epsilon^{1234} = 1$)[5]

$$\begin{aligned} \lim_{M\to\infty} \mathrm{tr}\gamma_5 f(\not{D}^2/M^2) &= \mathrm{tr}\gamma_5 \frac{1}{2!}\{\frac{ig}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\}^2 \int \frac{d^4k}{(2\pi)^4} f''(-k_\mu k^\mu) \\ &= \frac{g^2}{32\pi^2}\mathrm{tr}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \end{aligned} \tag{38}$$

When one combines (3.7) and (3.16), one reproduces the Atiyah-Singer index theorem (in continuum R^4 space).[7,8] We note that a local version of the index (anomaly) is valid for Abelian theory also. The global index (3.7) as well as a local version of the index (3.8) are both independent of the regulator $f(x)$ provided[5]

$$f(0) = 1, \quad f(\infty) = 0, \quad f'(x)x|_{x=0} = f'(x)x|_{x=\infty} = 0. \tag{39}$$

We have thus established that the lattice index in (3.7) for any algebraic relation in (1.2) is related to the Pontryagin index in a smooth continuum limit as

$$n_+ - n_- = \int d^4x \frac{g^2}{32\pi^2}\mathrm{tr}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \tag{40}$$

This shows that the instanton-related topological property is identical for all the algebraic relations in (1.2), and the Jacobian factor (3.6) in fact contains the correct chiral anomaly. (We are implicitly assuming that the index (3.7) does not change in the process of taking a continuum limit.)

A detailed perturbative analysis of chiral anomaly for the general operators with $k > 0$ has been performed, and the above result has been confirmed.[12] Also a numerical study of the index relation has been performed: The numerical result indicates the consistency of our analyses.[13]

4. Explicit construction of the lattice Dirac operator for $k > 1$

We now comment on an explicit construction of the lattice Dirac operator which satisfies the generalized algebraic relation (1.2) with $k > 0$. We start with the conventional Wilson fermion operator D_W defined by

$$D_W(x,y) \equiv i\gamma^\mu C_\mu(x,y) + B(x,y) - \frac{1}{a}m_0\delta_{x,y},$$

$$C_\mu(x,y) = \frac{1}{2a}[\delta_{x+\hat{\mu}a,y}U_\mu(y) - \delta_{x,y+\hat{\mu}a}U_\mu^\dagger(x)],$$
$$B(x,y) = \frac{r}{2a}\sum_\mu[2\delta_{x,y} - \delta_{y+\hat{\mu}a,x}U_\mu^\dagger(x) - \delta_{y,x+\hat{\mu}a}U_\mu(y)],$$
$$U_\mu(y) = \exp[iagA_\mu(y)], \tag{41}$$

where we added a constant mass term to D_W for later convenience. The parameter r stands for the Wilson parameter. Our matrix convention is that γ^μ are anti-hermitian, $(\gamma^\mu)^\dagger = -\gamma^\mu$, and thus $\not{C} \equiv \gamma^\mu C_\mu(n,m)$ is hermitian

$$\not{C}^\dagger = \not{C}. \tag{42}$$

The Dirac operator for a general value of k is constructed by rewriting (1.2) as a set of relations

$$H^{2k+1}\gamma_5 + \gamma_5 H^{2k+1} = 2H^{2(2k+1)},$$
$$H^2\gamma_5 - \gamma_5 H^2 = 0, \tag{43}$$

with $H = a\gamma_5 D$. The second relation in (4.3) is shown by using the defining relation (1.4), and the first of these relations (4.3) becomes identical to the ordinary Ginsparg-Wilson relation (1.1) if one defines $H_{(2k+1)} \equiv H^{2k+1}$. One can thus construct a solution to (4.3) by following the prescription used by Neuberger[2]

$$H_{(2k+1)} = \frac{1}{2}\gamma_5[1 + D_W^{(2k+1)}\frac{1}{\sqrt{(D_W^{(2k+1)})^\dagger D_W^{(2k+1)}}}] \tag{44}$$

where

$$D_W^{(2k+1)} \equiv i(\not{C})^{2k+1} + B^{2k+1} - (\frac{m_0}{a})^{2k+1} \tag{45}$$

The operator H itself is then finally defined by (in the representation where $H_{(2k+1)}$ is diagonal)

$$H = (H_{(2k+1)})^{1/2k+1} \tag{46}$$

in such a manner that the second relation of (4.3) is satisfied. This condition (4.3) is shown to be satisfied in the representation where $H_{(2k+1)}$ is diagonal.[12] Also the conditions $0 < m_0 < 2r = 2$ and

$$2m_0^{2k+1} = 1 \tag{47}$$

ensure the absence of species doublers and a proper normalization of the Dirac operator H.

5. Locality properties of general operators

We have explained that the general operators for any finite k give rise to correct chiral anomaly and index relations in the (naive) continuum limit. This suggests

that those operators are local for sufficiently smooth background gauge field configurations. The locality of the standard overlap operator with $k = 0$ has been established by Hernandez, Jansen and Lüscher,[15] and by Neuberger.[16]

As for the direct proof of locality of the operator D for general k, one can show it for the vanishing gauge field by using the explicit solution for the operator H in momentum representation[12,13]

$$\begin{aligned} H(ap_\mu) &= \gamma_5 (\frac{1}{2})^{\frac{k+1}{2k+1}} (\frac{1}{\sqrt{H_W^2}})^{\frac{k+1}{2k+1}} \{ (\sqrt{H_W^2} + M_k)^{\frac{k+1}{2k+1}} - (\sqrt{H_W^2} - M_k)^{\frac{k}{2k+1}} \frac{\not{s}}{a} \} \\ &= \gamma_5 (\frac{1}{2})^{\frac{k+1}{2k+1}} (\frac{1}{\sqrt{F_{(k)}}})^{\frac{k+1}{2k+1}} \{ (\sqrt{F_{(k)}} + \tilde{M}_k)^{\frac{k+1}{2k+1}} - (\sqrt{F_{(k)}} - \tilde{M}_k)^{\frac{k}{2k+1}} \not{s} \} \end{aligned} \tag{48}$$

where

$$\begin{aligned} F_{(k)} &= (s^2)^{2k+1} + \tilde{M}_k^2, \\ \tilde{M}_k &= [\sum_\mu (1 - c_\mu)]^{2k+1} - m_0^{2k+1} \end{aligned} \tag{49}$$

and

$$\begin{aligned} s_\mu &= \sin ap_\mu \\ c_\mu &= \cos ap_\mu \\ \not{s} &= \gamma^\mu \sin ap_\mu. \end{aligned} \tag{50}$$

For $k = 0$, this operator is reduced to Neuberger's overlap operator.[2] Here the inner product is defined to be $s^2 \geq 0$. This operator is shown to be free of species doublers for the parameter m_0 within the range $0 < m_0 < 2$ when we set $r = 1$, and $2m_0^{2k+1} = 1$ gives a proper normalization of H, namely, for an infinitesimal p_μ, i.e., for $|ap_\mu| \ll 1$,

$$H \simeq -\gamma_5 a \not{p}(1 + O(ap)^2) + \gamma_5 (\gamma_5 a \not{p})^{2k+2} \tag{51}$$

to be consistent with $H = \gamma_5 aD$; the last term in the right-hand side is the leading term of chiral symmetry breaking terms.

The locality of this explicit construction (5.1) is shown by studying the analytic properties in the Brillouin zone.[12] It is important to recognize that this operator is not ultra-local but exponentially local;[17] the operator $H(x, y)$ decays exponentially for large separation in coordinate representation

$$H(x, y) \sim \exp[-|x - y|/(2.5ka)]. \tag{52}$$

An explicit analysis of the locality of the operator $H_{(2k+1)}$ (*not* H itself)in the presence of gauge field, in particular, the locality domain for the gauge field strength $||F_{\mu\nu}||$ has been performed. The locality domain for $||F_{\mu\nu}||$ becomes smaller for larger k, but a definite non-zero domain has been established.[12] The remaining task is to show the locality domain of $||F_{\mu\nu}||$ for the operator $H = (H_{(2k+1)})^{1/(2k+1)}$. Due to the operation of taking the $(2k+1)$th root, an explicit analysis has not been performed yet, though a supporting argument has been given.[12]

6. Conclusion

We have reported the recent investigation of topological properties of a general class of lattice Dirac operators defined by the algebraic relation (1.2). All these operators satisfy the index theorem and thus they are topologically proper. A precise proof of the locality of these general Dirac operators with fully dynamical gauge fields remains to be formulated. The operators with large k is expected to exhibit infrared singularities in perturbative analyses as is suggested by the construction of $H_{(2k+1)}$ in (4.4), and thus the Wilsonian formulation of effective action, which is supposed to be free of infrared singularities, would be essential.

Although we discussed only 4-dimensional theory, the recent developments in the treatment of lattice fermions[11] may have some implications on 2-dimensional theory also, which is the main subject of this Symposium. In this respect, the fact that the lattice Dirac operators are not ultra-local but exponentially local[15] may be of some interest. See Ref. 18 for a Ginsparg-Wilson construction on a 2-dimensional fuzzy sphere.

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SPECTRAL FLOW OF HERMITIAN WILSON–DIRAC OPERATOR AND THE INDEX THEOREM IN ABELIAN GAUGE THEORIES ON FINITE LATTICES

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The spectral flows of the hermitian Wilson-Dirac operator for a continuous family of abelian gauge fields connecting different topological sectors are shown to have a characteristic structure leading to the lattice index theorem. The index of the overlap Dirac operator is shown to coincide with the topological charge for a wide class of gauge field configurations. It is also argued that in two dimensions the eigenvalue spectra for some special but nontrivial background gauge fields can be described by a set of universal polynomials and the index can be found exactly.

1. Introduction

For a long time it has been considered that chiral symmetries cannot be implemented on the lattice due to the Nielsen-Ninomiya no-go theorem. The situation, however, has completely changed after the discoveries of lattice Dirac operators satisfying the Ginsparg-Wilson (GW) relation.[1–3a] It is now possible to define exact chiral symmetry on the lattice.[6] Furthermore, the index theorem on the lattice relating the index of the GW Dirac operator to chiral anomaly has been obtained by Hasenfratz, Laliena and Niedermayer (HLN).[6,7]

In continuum theories the nontrivial topological structure of gauge fields are considered to be responsible for nonperturbative phenomena such as the large η-η' mass splitting in QCD and the fermion number violation in the standard model. The Atiyah-Singer (AS) index theorem provides a key relation there.

The HLN index theorem is known to reproduces the AS index theorem in the classical continuum limit.[8] However, it is not so clear what topological structure of the space of lattice gauge fields is related to the index of the GW Dirac operator on finite lattices. We expect that an extension of the index theorem relating the index directly to the topological invariants of gauge fields can be established also on finite lattices.

[a]For reviews, see Niedermeyer[4] and Lüscher.[5]

In this talk we would like to propose the "index theorem" relating the index of the GW Dirac operator directly to topological invariants of the lattice gauge fields,[9–12] mostly based on our recent results.[13] Working with the overlap Dirac operator[3] for compact U(1) theories on finite periodic lattices in two and four dimensions, we investigate the spectrum of the hermitian Wilson-Dirac operators numerically for a family of link variables connecting constant magnetic field configurations with distinct topological charges. Such an analysis has already been carried out by Narayanan and Neuberger[14] within the overlap formalism. They found the characteristic structure of the spectral flows leading to the index theorem. We will extend their analysis to include strong and nonsmooth gauge fields and find that the "index theorem" is kept intact for a wider class of gauge fields than those expected from the locality bounds.[15,16] The index and the topological charge, however, behave completely differently for nonsmooth gauge fields and the coincidence between the index and the topological charge breaks down. We also argue that the characteristic structure of the spectrum of the hermitian Wilson-Dirac operator in two dimensions can be understood exactly and the index can be computed for some special gauge field configurations.

2. The index theorem on the lattice

Let us begin with the definition of the overlap Dirac operator. On a $d = 2N$ dimensional euclidean hypercubic regular lattice it is defined by

$$D = 1 + \gamma_{d+1}\frac{H}{\sqrt{H^2}}\ , \tag{1}$$

where H is the hermitian Wilson-Dirac operator given by

$$H\psi(x) = \gamma_{d+1}\Bigg\{(d-m)\psi(x) - \sum_{\mu=1}^{d}\Bigg(\frac{1-\gamma_\mu}{2}U_\mu(x)\psi(x+\hat{\mu}) + \frac{1+\gamma_\mu}{2}U_\mu(x-\hat{\mu})\psi(x-\hat{\mu})\Bigg)\Bigg\}\ . \tag{2}$$

We have chosen the lattice spacing $a = 1$ and the Wilson parameter $r = 1$. The link variables $U_\mu(x)$ and the fermion wave function $\psi(x)$ are assumed to be periodic. The γ-matrices are taken to be hermitian and $\gamma_{d+1} = (-i)^{\frac{d}{2}}\gamma_1\cdots\gamma_d$ is employed. The m is chosen in the range $0 < m < 2$ to avoid species doubling and is taken to be $m = 1$ in our analysis.

Obviously, D is only well-defined for the gauge field configurations with $\det H \neq 0$. It may exhibit discontinuities at the link variables for which H has zero-modes. This can be seen by noting the relation between the index of D and the spectral asymmetry[17] of H

$$\mathrm{index}D = \mathrm{Tr}\gamma_{d+1}\left(1 - \frac{1}{2}D\right) = -\frac{1}{2}\mathrm{Tr}\frac{H}{\sqrt{H^2}} = -\frac{N_+ - N_-}{2}\ , \tag{3}$$

where N_+ (N_-) is the number of positive (negative) eigenvalues of H. Any two gauge field configurations with distinct indices cannot be continuously deformed into each other without crossing the configurations with $\det H = 0$. By excising such singular gauge field configurations the space of the link variables becomes disconnected. The HLN index theorem associates index$D$ to each connected component of the space of link variables.

Another way to give nontrivial topological structure to the space of link variables is to impose the conditions[9]

$$\sup_{x,\mu,\nu} ||1 - P_{\mu\nu}(x)|| \leq \eta \ , \tag{4}$$

where $P_{\mu\nu}$ is the standard plaquette variable and η (< 2) is a positive constant. The space of link variables becomes disconnected for sufficiently small η and it is possible to assign topological charge to each connected component.

In abelian theories the explicit forms of the topological charge[12,18] is given by

$$Q_N = \frac{1}{(4\pi)^N N!} \sum_x \epsilon_{\mu_1\nu_1\cdots\mu_N\nu_N} F_{\mu_1\nu_1}(x) F_{\mu_2\nu_2}(x+\hat{\mu}_1+\hat{\nu}_1) \times\cdots\times F_{\mu_N\nu_N}(x+\hat{\mu}_1+\hat{\nu}_1+\cdots+\hat{\mu}_{N-1}+\hat{\nu}_{N-1}) \ , \tag{5}$$

where $F_{\mu\nu} = -i \ln P_{\mu\nu}$ ($F_{\mu\nu}(x)| < \pi$) is the field strength and $\epsilon_{\mu_1\cdots\mu_d}$ is the Levi-Civita symbol in $d = 2N$ dimensions. The Q_N is a smooth function of the link variables within a connected component and takes an integer value given by[10,12]

$$Q_N = \frac{1}{2^N N!} \epsilon_{\mu_1\nu_1\cdots\mu_N\nu_N} m_{\mu_1\nu_1} m_{\mu_2\nu_2} \cdots m_{\mu_N\nu_N} \ , \tag{6}$$

where $2\pi m_{\mu\nu}$ is the magnetic flux through $\mu\nu$-plane. In fact it can be shown that the space of link variables of abelian gauge theory is decomposed into a finite number of connected components characterized by a set of integers $\{m_{\mu\nu}\}$, and any two configurations with the same set of magnetic fluxes can be continuously deformed into each other without violating (4).

In general one can find gauge field configurations that satisfy (4) and $\det H = 0$ simultaneously. If it happens, the index$D$ may jump within a connected component satisfying (4). However, if we choose η to satisfy

$$0 < \eta < \frac{2-\sqrt{2}}{d(d-1)} \ , \tag{7}$$

H cannot have zero-modes[15,16] and the index$D$ is a constant in each connected components satisfying (4). It is very natural to expect that the index and the topological charge coincide with each other. The precise form of the "index theorem" for abelian gauge theories can be stated as :

For the link variables satisfying (4) and (7) the index of D and the topological charge Q_N are related by

$$\text{index}D = (-1)^N Q_N \ . \tag{8}$$

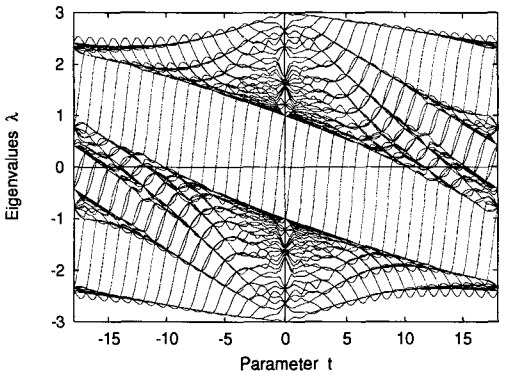

Fig. 1. Eigenvalue spectrum of H for $L = 6$ in two dimensions.

One way to find index$D$ is to count the net number of spectral flows of H as m varies from 0 to 1 while keeping the link variables. This approach was adopted by several authors.[14,19,20] To see how the index of D behaves as one varies the link variables continuously from one connected component to another it is more convenient to investigate spectral flows of H for a continuous family of link variables connecting configurations with constant field strengths $F_{\mu\nu}(x) = 2\pi m_{\mu\nu}/L^2$. Concretely, we take the one-parameter family of link variables in two dimensions

$$U_1^{(t)}(x) = \exp\left[-it\frac{2\pi}{L}x_2\delta_{x_1,L-1}\right], \quad U_2^{(t)}(x) = \exp\left[it\frac{2\pi}{L^2}x_1\right]. \quad (0 \le x_\mu < L) \tag{9}$$

The field strength is a constant for any value of t except for the corner point $x_{1,2} = L - 1$. As a function of t, the topological charge Q_1 has discontinuities at $t = \dfrac{(2j+1)L^2}{2(L^2-1)} \approx j + \dfrac{1}{2}$ $(j = 0, \pm 1, \pm 2, \cdots)$.

3. Numerical results in two and four dimensions

We have numerically analyzed the eigenvalue spectrum of H over the range $-L^2/2 \le t \le L^2/2$ for $2 \le L \le 15$. It is helpful to note the following facts: (1) In two dimensions the eigenvalues λ of H are bounded by $|\lambda| \le 3$.[16] (2) The eigenvalue spectrum of H is L^2 periodic in t. (3) The eigenvalue spectrum of H is symmetric with respect to the point $t = 0$. Hence index$D$ is an L^2 periodic odd function of t and vanishes at $t = 0, \ \pm L^2/2$, where the spectrum is symmetric.

In Figure 1 the whole spectrum of H is shown for $L = 6$. It is consistent with the result obtained by Narayanan and neuberger.[14] The characteristic structure of the spectrum is not changed with L. We find that most of the eigenvalues lie within the upper and the lower trapezoid regions symmetrically separated by the parallelogram region and there are large gaps at integer t. As t increases by unity from an integer, an eigenvalue belonging to the lower trapezoid crosses the parallelogram upward

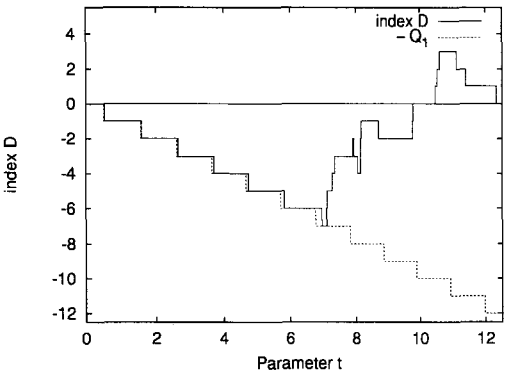

Fig. 2. The index$D$ and $-Q_1$ are plotted for $0 \leq t \leq L^2/2$ ($L = 5$, $d = 2$).

and moves to the upper trapezoid. In particular on the interval of the horizontal axis cut by the parallelogram eigenvalues change the sign for a sequence of the values of t, where the index jumps by -1.

The index$D$ and $-Q_1$ are plotted over the region $0 \leq t \leq L^2/2$ for $L = 5$ in Figure 2. The correspondence between the index and the topological charge is very excellent for $0 \leq t \leq L^2/4$ except for the points around the discontinuities. This is consistent with the numerical results given by Chiu.[21]

For $t > L^2/4$ the index$D$ and $-Q_1$ behaves completely differently. Such can be considered as a kind of lattice specific phenomena. It is safer to avoid such configurations in order to keep proper connection with continuum theories.

We have also carried out a similar analysis in four dimensions by taking the link variables (9) in the presence of a constant magnetic flux through 34-plane. The characteristic features of the spectral flows observed in two dimensions can be seen also in four dimensions. In Figure 3 we indicate the spectrum of H for $L = 4$, $|\lambda| \leq 1$, $m_{34} = 1$ and $|t| \leq 8$. A well-isolated eigenvalue crosses the horizontal axis, where index$D$ increases by one unit. The characteristic feature of the spectrum is not changed for $m_{34} > 1$. In general m_{34} adjacent eigenvalues flow downward and index$D$ increases by m_{34} when they cross the t-axis. This is consistent with (6).

Though our analysis is restricted to rather small lattice sizes $2 \leq L \leq 4$, we anticipate that the parallelogram region expands rapidly enough as L increase and the coincidence between the index and the topological charge occurs for a wide class of gauge field configurations. Incidentally, for gauge fields satisfying (4) and (7) nonvanishing topological charges can be realized only for $L \geq 9$.

4. Some exact results in two dimensions

Coming back to two dimensions, we now show that the characteristic structure of the spectrum of H can be understood by noting that the eigenvalue equations for constant field strength configurations can be converted to equivalent simple one-

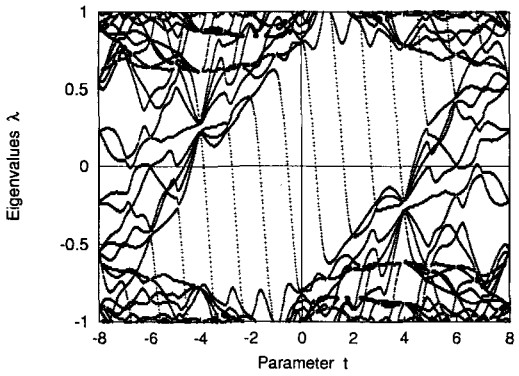

Fig. 3. Eigenvalue spectrum of H for $|\lambda| \leq 1$, $s = 1$ and $|t| \leq 8$ ($L = 4$ and $d = 4$).

dimensional systems. Here we will consider the cases of field configurations at $t = r$ and $t = L^2/r = sL$, where r and s are arbitrary positive integers satisfying $L = rs$.

At $t = sL$ the H is independent of x_2 and r periodic in x_1. This implies that the system can be converted to a smaller one with degrees of freedom $2r$ by Fourier transformations. It can be shown that the eigenvalues λ of H at $t = sL$ satisfy the secular equation

$$\det \begin{pmatrix} B(p,q) - \lambda & C(p,q) \\ C(p,q)^\dagger & -B(p,q) - \lambda \end{pmatrix} = 0 \,, \tag{10}$$

where p and q are the Fourier momenta

$$p = \frac{2\pi k}{L} \,, \quad q = \frac{2r\pi l}{L} \,, \qquad (0 \leq k < L \,, \quad 0 \leq l < s) \tag{11}$$

and $B(p,q)$, $C(p,q)$ are $r \times r$ matrices defined by

$$(B(p,q))_{kl} = -\frac{1}{2}\delta^{(q)}_{k,l+1} + \left[1 - \cos\left(p + \frac{2\pi k}{r}\right)\right]\delta^{(q)}_{k,l} - \frac{1}{2}\delta^{(q)}_{k+1,l} \,,$$

$$(C(p,q))_{kl} = -\frac{1}{2}\delta^{(q)}_{k,l+1} + \sin\left(p + \frac{2\pi k}{r}\right)\delta^{(q)}_{k,l} + \frac{1}{2}\delta^{(q)}_{k+1,l} \,. \quad (0 \leq k,l < r) \tag{12}$$

The $\delta^{(q)}_{k,l}$ is the Kronecker's δ-symbol for $0 \leq k,l \leq r-1$ and satisfies the twisted boundary conditions $\delta^{(q)}_{k,r} = \mathrm{e}^{-iq}\delta_{k,0}$ and $\delta^{(q)}_{r,k} = \mathrm{e}^{iq}\delta_{0,k}$.

The secular equation (10) can be rewritten in the following form

$$f_r(\lambda) = \frac{(-1)^{r-1}}{2^{r-4}} \sin^2\frac{rp}{2} \sin^2\frac{q}{2} \,, \tag{13}$$

where $f_r = \lambda^{2r} + \cdots$ is a polynomial of degree $2r$ and is independent of p and q. For $1 \leq r \leq 4$ it is explicitly given by

$$f_1 = \lambda^2 - 1 \,,$$

$$f_2 = \lambda^4 - 6\lambda^2 + 1 \ ,$$
$$f_3 = \lambda^6 - 9\lambda^4 + \frac{69}{4}\lambda^2 - 3\sqrt{3}\lambda - \frac{1}{4} \ ,$$
$$f_4 = \lambda^8 - 12\lambda^6 + 42\lambda^4 - 8\lambda^3 - 44\lambda^2 + 24\lambda - 3 \ . \tag{14}$$

In particular $r = 1$ corresponds to the free spectrum since the link variables become trivial. The eigenvalues λ must satisfy the inequality

$$0 \leq (-1)^{r-1} 2^{r-4} f_r(\lambda) \leq 1 \ . \tag{15}$$

This gives rise to $2r$ allowed narrow intervals for λ. In each interval there appears exactly $L \times s$ eigenvalues. As an example, we consider the case that $L = 4s$ and take $r = 4$. For any p, q the eight roots of (13) are separately located in well-separated eight narrow intervals. There are three negative eigenvalues and five positive eigenvalues for each p, q. Then the index is then given by $-1 \times L \times s = -sL$. This coincides with $-Q_1$. In general $f_r(\lambda) = 0$ has $r-1$ negative roots and $r+1$ positive roots for $r \geq 4$ and, hence, index$D = -sL = -Q_1$ at $t = sL$. On the other hand it is vanishing for $1 \leq r \leq 3$ since the number of positive roots and that of negative ones are always equal.

It is possible to extend the above arguments to integers $t = r$ $(\leq L)$. In particular we arrive at the relation

$$\det(H - \lambda)|_{t=r} = (f_{sL}(\lambda))^r \ . \tag{16}$$

This gives at $t = r$ the index $-r = -Q_1$ for $sL \geq 4$ and 0 for $sL \leq 3$. These are completely consistent with the numerical results.

5. Summary

We have confirmed the equality (8) between the index and the topological charge for abelian gauge theories on finite periodic lattices in two and four dimensions. It holds true for a wider class of gauge field configurations than those satisfying the locality bounds.[15,16] The condition (4) with (7) excludes uniformly the configurations for which the discrepancy between the index and the topological charge appears, ensuring the index theorem (8).

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DIMERS AND SPANNING TREES: SOME RECENT RESULTS

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This paper reviews some recent progress on dimer and spanning tree enumerations. We use the Kasteleyn formulation to enumerate close-packed dimers on a simple-quartic net embedded on non-orientable surfaces, and obtain solutions in the form of double products. For spanning trees the enumeration is carried out by evaluating the eigenvalues of the Laplacian matrix associated with the lattice, a procedure which holds in any spatial dimension. In two dimensions a bijection due to Temperley relates spanning tree and dimer configurations on two related lattices. We use this bijection to enumerate dimers on a net with a vacancy on the boundary. It is found that the occurrence of a vacancy induces a $\sqrt{N}$ correction to the enumeration, where N is the linear size of the lattice, and changes the central charge from $c = 1$ to -2.

1. Introduction

Theoretical studies of the physics of real systems often lead to problems of far-reaching interests in mathematics, and solutions to the mathematical problems in turn yield new insights to the physical problems. One such example is the advent of dimer statistics, a subject matter at the forefront of mathematical research, from the evaluation of the adsorption entropy of diatomic molecules on a surface.[1] Another example is the arising of the notion of spanning trees, again a subject matter of immense interest in graph theory, from the theory of electric network currents.[2] In two dimensions these two mathematical problems are further interrelated, a fact recognized again through the consideration of the physics of the problems.[3] In this paper we describe and review some recent progress on dimers and spanning trees obtained by the author and co-workers.[4–8]

We first define the problems of dimer and spanning tree enumerations. The dimer problem is treated in section 2 where the Kasteleyn formulation is outlined and used to obtain the dimer generating function for two non-orientable surfaces, the Möbius strip and the Klein bottle. In section 3 an established result in graph theory is used to enumerate spanning trees. In section 4 we describe a bijection due to Temperley[3] which relates dimers and spanning trees on two related lattices, and use it to establish the independence of the dimer generating function on the location of a vacancy on the boundary of a simple-quartic net. The bijection also

leads to an explicit evaluation of the dimer generating function for the defect lattice. Results of finite-size analyses, which lead to corrections induced by the geometry of a boundary vacancy, are presented in section 5.

2. Definitions

We shall consider a regular lattice $\mathcal{L}$ having a vertex (site) set V and edge set E, but much of our results are applicable more generally to $\mathcal{L}$ being an arbitrary graph. Number the sites from 1 to $|V|$ and associate to the edge e_{ij} connecting vertices i and j a weight x_{ij}, with $x_{ij} = 0$ if there is no edge connecting i and j. A dimer covering $\mathcal{P}$ (for $|V|$ = even) is a pairing of the $|V|$ vertices into $|V|/2$ pairs. We say that the edge e_{ij} is covered by a dimer if e_{ij} appears in $\mathcal{P}$. Then, the dimer generating function is

$$Z(\mathcal{L}; \{x_{ij}\}) = \sum_{\mathcal{P}} \prod_{e_{ij} \in \mathcal{P}} x_{ij}, \tag{1}$$

where the summation is taken over all dimer coverings. The total number of dimer coverings is obtained by setting $x_{ij} = 1$, or

$$N_{\text{dimer}}(\mathcal{L}) = Z(\mathcal{L}; 1). \tag{2}$$

Specializing (1) to an $\mathcal{M} \times \mathcal{N}$ simple-quartic lattice of $\mathcal{M}$ rows and $\mathcal{N}$ columns with edge weights z_h and z_v respectively in the horizontal and vertical directions, the dimer generating function is

$$Z_{\mathcal{M},\mathcal{N}}(\mathcal{L}; z_h, z_v) = \sum_{\mathcal{P}} z_h^{n_h} z_v^{n_v}, \tag{3}$$

where n_h and n_v are, respectively, the numbers of horizontal and vertical dimers in $\mathcal{P}$.

Next we define spanning trees. A subset of edges $T \subset E$ is a spanning tree if it has $|V| - 1$ edges with at least one edge incident at each vertex. Thus T has no cycles. The enumeration of spanning trees concerns with the evaluation of the spanning tree generating function

$$T(\mathcal{L}; \{x_{ij}\}) = \sum_{T \subset E} \prod_{e_{ij} \in T} x_{ij}, \tag{4}$$

where the summation is taken over all spanning trees configurations T. Particularly, the total number of spanning trees on $\mathcal{L}$ is obtained by setting $x_{ij} = 1$, or

$$N_{\text{SPT}}(\mathcal{L}) = T(\mathcal{L}; 1). \tag{5}$$

Specializing (5) to a simple-quartic net as in the above, the tree generating function assumes the form

$$T_{\mathcal{M},\mathcal{N}}(\mathcal{L}; z_h, z_v) = \sum_{T} z_h^{n_h} z_v^{n_v}, \tag{6}$$

where n_h and n_v are, respectively, the numbers of horizontal and vertical edges in T.

The evaluation of (3) for $\mathcal{L}$ with free and toroidal boundary conditions was first accomplished by Kasteleyn[9] and Temperley and Fisher.[10,11] Here we extend the solution to non-orientable surfaces,[4,7,12] and to the net $\mathcal{L}$ having a boundary vacancy.[8]

3. Dimer enumerations

3.1. *Kasteleyn formulation*

It is an elementary fact that the superposition of two dimer configurations decomposes a lattice $\mathcal{L}$ into *superposition* polygons, namely, polygons formed by tracing along dimers from vertex to vertex. Orient all edges of $\mathcal{L}$ and define an $|V| \times |V|$ antisymmetric matrix $\mathbf{A}(x_{ij})$ with elements

$$A_{ij} = -A_{ji} = x_{ij}, \qquad \text{if the edge } ij \text{ is directed from } i \text{ to } j. \tag{7}$$

Then, Kasteleyn[9] has established the remarkable result that

$$Z(\mathcal{L}; \{x_{ij}\}) = \sqrt{|\mathbf{A}(x_{ij})|}, \tag{8}$$

where $|\mathbf{A}|$ is the determinant of $\mathbf{A}$, provided that lattice edges are oriented such that the product of all edge weights around *every* possible superposition polygon is negative. Namely,

$$x_{ij}x_{jk}\cdots x_{\ell i} < 0 \qquad \text{(the Kasteleyn criterion)} \tag{9}$$

for sites $i, j, k, ..., \ell$ around a superposition polygon arranged in the order of, say, a clockwise (cw) direction. Henceforth we shall refer to the sign of $-x_{ij}x_{jk}\cdots x_{\ell i}$ as the sign of the (superposition) polygon.

The Kasteleyn criterion (9) is remarkable since it says nothing about the dimensionality of the lattice. It is this flexibility which permits its application to non-orientable surfaces. However, even if the Kasteleyn criterion is met, it still remains to evaluate the determinant $|\mathbf{A}(x_{ij})|$ which can be a formidable task in some cases.

3.2. *Simple-quartic nets on non-orientable surfaces*

We consider the enumeration of dimers on an $\mathcal{M} \times \mathcal{N}$ simple-quartic net embedded on two non-orientable surfaces. The net forms a Möbius strip if there is a twisted (Möbius) boundary condition in the horizontal direction as shown in Fig. 1, and a Klein bottle if, in addition to the twisted boundary condition, there is also a periodic boundary condition in the vertical direction.

To satisfy the Kasteleyn criterion (9) we make use of a trick due to T. T. Wu[13] of associating a factor i to dimer weights in one spatial direction. For a simple-quartic net with free boundaries, one associates a factor i to dimer weights in the

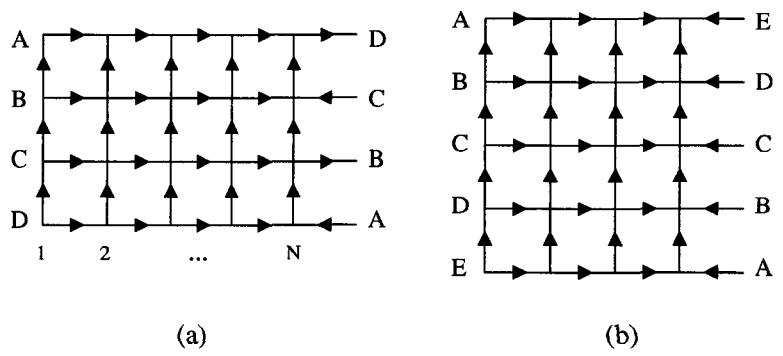


Fig. 1. An $\mathcal{M} \times \mathcal{N}$ Möbius strip and the associated edge orientations. A, B, C, D, E are repeated sites. (a) $(\mathcal{M},\mathcal{N}) = (5,4)$. (b) $(\mathcal{M},\mathcal{N}) = (4,5)$.

direction in which the number of lattice sites is odd. For $\{\mathcal{M},\mathcal{N}\} = \{\text{even},\text{odd}\}$, for example, one replaces z_h by iz_h. If the number of lattice sites is even in both directions, then the factor i can be associated to dimers in either direction.

Next one orients all parallel lattice edges uniformly in the same direction. To see that this orientation satisfies the Kasteleyn criterion for free boundary conditions, the case considered by Wu, one superimposes any dimer covering C_1 with a standard C_0 in which the lattice is covered only by parallel dimers with real weights. Then, each superposition polygon formed by C_1 and C_0 contains an even number of arrows pointing in each (cw or ccw) direction as well as a factor $i^{4n+2} = -1$, where n is a nonnegative integer. It follows that the criterion (9) holds and the sign of every polygon is positive. Note that the construction of C_0 requires either $\mathcal{M}$ or $\mathcal{N}$ to be even.

On non-orientable surfaces a superposition polygon can wrap around the lattice in the horizontal direction, and this causes problems in realizing the Kasteleyn criterion. Particularly, one needs to pay attention to whether $\mathcal{M}$ and $\mathcal{N}$ are even or odd. Starting from a net with free boundaries, there are $\mathcal{M}$ additional horizontal "connecting" edges (shown in Fig. 1) and, in the case of the Klein bottle, $\mathcal{N}$ additional vertical connecting edges (not shown in Fig. 1). These edges need to be oriented. It turns out that, except in the case that both $\mathcal{M},\mathcal{N}$ are even,[4] it is not possible to orient the connecting edges so that the Kasteleyn criterion holds for all polygons wrapping around the lattice. However, one can take advantage of the regularity of the signs of the polygons to extract the desired solution.

Let the horizontal connecting edges carry a weight z, and let $\mathbf{A}(z)$ denote the resulting antisymmetric matrix (7). It is straightforward to show[7] that superposition polygons containing $4n$ and $4n+1$ connecting z edges in its perimeter, where n is an integer, have the same sign and those having $4n+2$ and $4n+3$ connecting z edges have the opposite sign. It follows that[7] the desired dimer generating function

on a net (with uniform weights z_h and z_v) is given by the linear combination

$$Z_{\mathcal{M},\mathcal{N}}(\mathcal{L}; z_h, z_v) = \frac{1}{2}\left[(1-i)\sqrt{|\mathbf{A}(iz_h)|} + (1+i)\sqrt{|\mathbf{A}(-iz_h)|}\right]$$
$$= \mathrm{Re}\left[(1-i)\sqrt{|\mathbf{A}(iz_h)|}\right] \tag{10}$$

where Re denotes the real part. This formula applies to both the Möbius strip and the Klein bottle.

The determinant $|\mathbf{A}(\pm iz_h)|$ can be evaluated by computing the eigenvalues of the matrix $\mathbf{A}(\pm iz_h)$, and the algebra is somewhat different for the two lattices shown in Fig. 1. We refer to Ref. 7 for details and give here only the final result,

$$Z_{\mathcal{M},\mathcal{N}}(\mathcal{L}; z_h, z_v)$$
$$= z_h^{\mathcal{M}\mathcal{N}/2}\mathrm{Re}\left[(1-i)\prod_{m=1}^{[(\mathcal{M}+1)/2]}\prod_{n=1}^{\mathcal{N}}\left(2i(-1)^{\frac{\mathcal{M}}{2}+m+1}\sin\frac{(4n-1)\pi}{2\mathcal{N}} + 2\,X_m\right)\right], \tag{11}$$

where $[x]$ is the integral part of x, and

$$X_m = \left(\frac{z_v}{z_h}\right)\cos\frac{m\pi}{\mathcal{M}+1} \qquad \text{for the Moebius strip}$$
$$= \left(\frac{z_h}{z_v}\right)\sin\frac{(2m-1)\pi}{\mathcal{M}} \qquad \text{for the Klein bottle.} \tag{12}$$

For $\mathcal{N}$ = even, the product in (11) is real and (11) reduces to a simpler form

$$Z_{\mathcal{M},\mathcal{N}}(\mathcal{L}; z_h, z_v) = z_h^{\mathcal{M}\mathcal{N}/2}\prod_{m=1}^{[(\mathcal{M}+1)/2]}\prod_{n=1}^{\mathcal{N}/2}\left(4\sin^2\frac{(4n-1)\pi}{2\mathcal{N}} + 4\,X_m^2\right). \tag{13}$$

For the Möbius strip Tesler[12] has also obtained the solution in terms of generalized Fibonacci numbers. It can be shown that his solutions are the same as those given by (11).

In all cases, one obtains in the thermodynamic limit the per-site bulk free energy

$$f_{\mathrm{dimer}}(z_h, z_v) \equiv \lim_{\mathcal{M},\mathcal{N}\to\infty}\frac{1}{\mathcal{M}\mathcal{N}}\ln Z_{\mathcal{M},\mathcal{N}}(\mathcal{L}; z_h, z_v)$$
$$= \frac{1}{4\pi^2}\int_0^\pi d\theta\int_0^\pi d\phi\ln\left[4\,z_h^2\sin^2\theta + 4\,z_v^2\sin^2\phi\right]. \tag{14}$$

This gives the per-site entropy of the adsorption of diatomic molecules on a simple-quartic lattice as

$$f_{\mathrm{dimer}}(1,1) = \frac{G}{\pi} \tag{15}$$

where G is the Catalan constant given by

$$G = 1 - 3^{-2} + 5^{-2} - 7^{-2} + \cdots = 0.915\,965\,594\ldots. \tag{16}$$

4. Spanning tree enumerations

4.1. *The Laplacian matrix*

First we recall an established result in graph theory on spanning trees. Consider a graph $\mathcal{L}$ with edge weights x_{ij}. In analogous to (7) we define a $|V| \times |V|$ symmetric matrix $\mathbf{B}$ with elements

$$\begin{aligned} B_{ij} = B_{ji} = x_{ij} \qquad & \text{if } i \text{ and } j \text{ are adjacent} \\ = 0 \qquad & \text{otherwise.} \end{aligned} \tag{17}$$

Thus $\mathbf{B}$ is the adjacency matrix of $\mathcal{L}$ if $x_{ij} = 1$. Define further a diagonal matrix $\mathbf{\Delta}$ with elements

$$\Delta_{ii} = \sum_{j \neq i} x_{ij}. \tag{18}$$

Then, the matrix

$$\mathbf{Q} = \mathbf{\Delta} - \mathbf{B} \tag{19}$$

is the Laplacian of the lattice $\mathcal{L}$. The Laplacian matrix has the property that the sum of each row or column is zero, so it has a zero eigenvalue.

Let $\lambda_1, \lambda_2, ..., \lambda_{|V|-1}$ be the $|V| - 1$ nonzero eigenvalues of $\mathcal{L}$. Two fundamental theorems in graph theory[14,15] state that we have

$$T(\mathcal{L}; \{x_{ij}\}) = \text{any cofactor of } \mathbf{Q} \tag{20}$$

$$= \frac{1}{|V|} \prod_{i=1}^{|V|-1} \lambda_i \, . \tag{21}$$

A proof of (20) can be found in any book of graph theory (see, e.g., Ref. 14), and an elementary proof of the equivalence of (20) and (21) has been given in Ref. 5.

4.2. *Simple-quartic lattice*

We have used the formulation (21) to derive spanning tree generating functions for finite hypercubic lattices in d dimensions under various boundary conditions[5] as well as for regular lattices in two dimensions.[6] Here we discuss only the simple-quartic lattice with free boundary conditions which is relevant to our ensuing discussions.

For the simple-quartic lattice with free boundary conditions the Laplacian assumes the form

$$\mathbf{Q} = 2(z_h + z_v) I_{\mathcal{M}} \otimes I_{\mathcal{N}} - z_h H_{\mathcal{M}} \otimes I_{\mathcal{N}} - z_v I_{\mathcal{M}} \otimes H_{\mathcal{N}} \tag{22}$$

where I_N is an $N \times N$ identity matrix, and H_N is the $N \times N$ tri-diagonal matrix

$$H_N = \begin{pmatrix} 1\,1\,0\,0\cdots 0\,0\,0 \\ 1\,0\,1\,0\cdots 0\,0\,0 \\ 0\,1\,0\,1\cdots 0\,0\,0 \\ \vdots\,\vdots\,\vdots\,\vdots\,\ddots\,\vdots\,\vdots\,\vdots \\ 0\,0\,0\,0\cdots 1\,0\,1 \\ 0\,0\,0\,0\cdots 0\,1\,1 \end{pmatrix}. \tag{23}$$

The eigenvalues of H_N are[5]

$$\lambda_n = 2\cos\frac{n\pi}{N}, \quad n = 0, 1, \ldots, N-1. \tag{24}$$

Then, by diagonalizing $\mathbf{Q}$ in the two subspaces of dimensions M and N separately, one obtains the its eigenvalues

$$\lambda_{mn} = 2z_h\left[1-\cos\frac{m\pi}{\mathcal{M}}\right] + 2z_v\left[1-\cos\frac{n\pi}{\mathcal{N}}\right], \quad m = 0, 1, \ldots, \mathcal{M}-1, \quad n = 0, 1, \ldots, \mathcal{N}-1. \tag{25}$$

Using (21), one then obtains

$$T_{\mathcal{M},\mathcal{N}}(\mathcal{L}; z_h, z_v) = \frac{1}{\mathcal{M}\mathcal{N}} \prod_{m=0}^{\mathcal{M}-1} \prod_{n=0}^{\mathcal{N}-1} \left[4\, z_h \sin^2\frac{m\pi}{2\mathcal{M}} + 4\, z_v \sin^2\frac{n\pi}{2\mathcal{N}}\right], \quad (m,n) \neq (0,0) \tag{26}$$

and the per-site free energy

$$\begin{aligned} f_{\mathrm{SPT}}(z_h, z_v) &\equiv \lim_{\mathcal{M},\mathcal{N}\to\infty} \frac{1}{\mathcal{M}\mathcal{N}} \ln T_{\mathcal{M},\mathcal{N}}(\mathcal{L}; z_h, z_v) \\ &= \frac{1}{\pi^2}\int_0^\pi d\theta \int_0^\pi d\phi \ln\left[4\, z_h \sin^2\theta + 4\, z_v \sin^2\phi\right]. \end{aligned} \tag{27}$$

This leads to

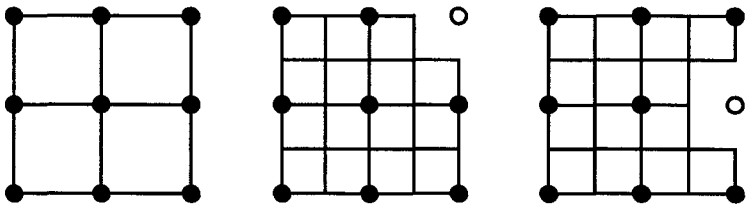
$$f_{\mathrm{SPT}}(1,1) = \frac{4}{\pi}G. \tag{28}$$

5. Simple-quartic net with a vacancy

The similarity between (14) and (27) is striking, since it suggests a connection between the dimer and spanning tree problems. Indeed, Temperley[3] has found a bijection between dimer and spanning tree configurations on two related lattices. The bijection has recently been extended to general planar graphs with weighted and/or oriented edges.[16] Here, we describe a version of the bijection relevant to out discussions.[8]

5.1. *Temperley bijection*

Starting from an $\mathcal{M} \times \mathcal{N}$ simple-quartic net $\mathcal{L}$ with free boundaries, one constructs a dimer lattice $\mathcal{L}_D$ by i) adding a new site at the midpoint of each edge of $\mathcal{L}$, ii) inserting in each internal face of $\mathcal{L}$ a new site connected to the midpoints of the 4 edges of $\mathcal{L}$ surrounding it, and iii) removing one of the original boundary sites of $\mathcal{L}$ together with its edges incident from the neighboring midpoints inserted in i). Thus, $\mathcal{L}_D$ has a total of $(2\mathcal{M}-1)(2\mathcal{N}-1)-1$ sites consisting of the original $\mathcal{M}\mathcal{N}-1$ sites of $\mathcal{L}$ and the remaining $(2\mathcal{M}-1)(2\mathcal{N}-1)-\mathcal{M}\mathcal{N}$ new sites. Examples of constructing $\mathcal{L}_D$ for $\mathcal{M}=\mathcal{N}=3$ are shown in Fig. 2. Then we have the following Temperley bijection as contained in results reported in Ref. 16.



(a) (b) (c)

Fig. 2. (a) A spanning tree lattice $\mathcal{L}$. (b) A dimer lattice $\mathcal{L}_D$ constructed from $\mathcal{L}$ with one corner site removed. (c) A dimer lattice $\mathcal{L}_D$ constructed from $\mathcal{L}$ with one boundary site removed. Open circles denote removed sites.

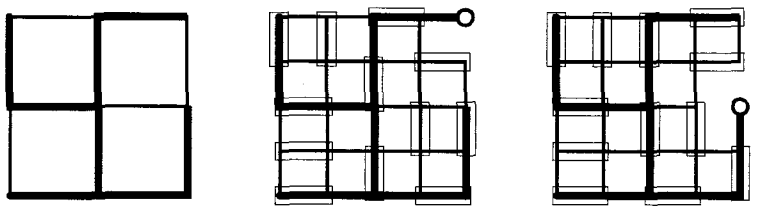
Temperley bijection: There exists a one-one correspondence between spanning tree configurations on $\mathcal{L}$ and dimer configurations on $\mathcal{L}_D$.

To see that the bijection holds, one observes that to each spanning tree configuration on $\mathcal{L}$, one can construct a unique dimer configuration on $\mathcal{L}_D$ by first laying a dimer along each tree edge, starting from the edge(s) covering the site of $\mathcal{L}_D$ which has (have) been removed, and proceed along the spanning tree edges in an obvious fashion. After laying dimers along all tree edges, the remaining sites of $\mathcal{L}_D$ can then be covered by dimers in a unique way.[3] Conversely, starting from each dimer configuration on $\mathcal{L}_D$, one constructs a unique tree configuration on $\mathcal{L}$ by drawing bonds (tree edges) along dimers originating from all odd sites. These bonds cannot form close circuits, since otherwise they would have enclosed an odd number of sites of $\mathcal{L}_D$ which is not permitted in close-packed dimer configurations. This process leads to a unique tree configuration on $\mathcal{L}$. This completes the proof.

Examples of the Temperley bijection are shown in Fig. 3.

5.2. *Dimer enumeration on $\mathcal{L}_D$*

To enumerate dimers on $\mathcal{L}_D$ which is a simple-quartic lattice with a defect on its boundary, one must start from $\mathcal{M},\mathcal{N}$ odd so that the net of $(2\mathcal{M}-1)(2\mathcal{N}-1)-1$ sites admits dimer coverings. Let $\mathcal{L}_D$ and $\mathcal{L}'_D$ be two dimer lattices derived from



(a) (b) (c)

Fig. 3. Bijections between a spanning tree configuration on $\mathcal{L}$ shown in (a) and dimer configurations on two $\mathcal{L}_D$ lattices shown in (b) and (c).

$\mathcal{L}$ as described in the above. Let the dimer and spanning tree edge weights be the same z_h and z_v. Then, the Temperley bijection implies a one-one correspondence between dimer configurations on $\mathcal{L}_D$ and $\mathcal{L}'_D$ via the mutual equivalence to spanning trees. A moment's reflection now shows that the two dimer configurations also have identical weights.[7] It follows that we have the identity

$$Z(\mathcal{L}_D; z_h, z_v) = Z(\mathcal{L}'_D; z_h, z_v). \tag{29}$$

Namely, the dimer generating function is independent of the location of the boundary vacancy. This is a somewhat unexpected result which is difficult to see without the use of the Temperley bijection.

For $\mathcal{M}$,$\mathcal{N}$ odd, the aforementioned scheme of orienting lattice edges for the realization of the Kasteleyn criterion no longer holds, since in constructing C_1 one needs either $\mathcal{M}$ or $\mathcal{N}$ be even. However, using the Temperley bijection it can be shown[8] that the dimer generating function for $\mathcal{L}_D$ is given by

$$Z(\mathcal{L}_D; z_h, z_v) = z_h^{\mathcal{M}(\mathcal{N}-1)} z_v^{\mathcal{N}(\mathcal{M}-1)} T_{\mathcal{M},\mathcal{N}}\left(\mathcal{L}; \frac{z_h}{z_v}, \frac{z_v}{z_h}\right). \tag{30}$$

Particularly, for $z_h = z_v = 1$, one has

$$\begin{aligned} N_{\text{dimer}}(\mathcal{L}_D) &= Z(\mathcal{L}_D; 1, 1) = N_{\text{SPT}}(\mathcal{L}) \\ &= \frac{1}{\mathcal{M}\mathcal{N}} \prod_{m=0}^{\mathcal{M}-1} \prod_{n=0}^{\mathcal{N}-1} \left[4\sin^2 \frac{m\pi}{2\mathcal{M}} + 4\sin^2 \frac{n\pi}{2\mathcal{N}}\right], \quad (m,n) \neq (0,0), \end{aligned} \tag{31}$$

where the last line is obtained from (26) with $z_h = z_v = 1$. However, one must note that $\mathcal{L}_D$ is a $(2\mathcal{M}-1) \times (2\mathcal{N}-1)$ lattice with a boundary defect.

6. Finite-size analyses

Finite-size expansions of physical quantities associated with two-dimensional lattice models have been of interest both in physics[4,17] and in mathematics.[18] Let $Z_{M,N}$

denote the partition function of a lattice model on an $M \times N$ lattice. For large M, N one has the expansion

$$\ln Z_{M,N} = MNf_{\text{bulk}} + Nc_1 + Mc_2 + c_3 + o(1), \tag{32}$$

where f_{bulk} is the bulk free energy as computed in (14) and (27), c_1, c_2 are constants independent of M and N, and c_3 is a constant which can depend on M and N. In conformal field theory one further computes the limits

$$\frac{1}{M}\lim_{N\to\infty}\frac{\ln Z_{M,N}}{N} = f_{\text{bulk}} + \frac{c_1}{M} + \frac{\Delta_1}{M^2} + o(M^{-2}); \tag{33}$$

$$\frac{1}{N}\lim_{M\to\infty}\frac{\ln Z_{M,N}}{M} = f_{\text{bulk}} + \frac{c_2}{N} + \frac{\Delta_2}{N^2} + o(N^{-2}), \tag{34}$$

using which the central charge c can be computed from the values of Δ_1 and Δ_2. These expansions hold for general M and N regardless whether they are even or odd.

We have carried out finite-size analyses of our solutions for dimer[4] and spanning tree[8] solutions. Here we give the results on the number of dimer configurations ($z_h = z_v = 1$) (see also Refs. 11, 19, 20 for equivalent results).

For an $M \times N$ dimer lattice with $MN =$ even we use (13) for the number of dimer configurations $N_{\text{dimer}} = Z_{M,N}(\mathcal{L}; 1, 1)$ and obtain

$$\begin{aligned} f_{\text{bulk}} &= \frac{G}{\pi}, \\ c_1 = c_2 &= \frac{G}{\pi} - \frac{1}{2}\ln(1+\sqrt{2}) \\ c_3 &= \frac{G}{\pi} + \frac{3}{4}\ln 2 - \ln(1+\sqrt{2}) + \frac{\pi M}{24N} + \sum_{n=1}^{\infty}\ln\left(1 + e^{-(2n-1)\pi M/N}\right). \end{aligned} \tag{35}$$

Despite its apparent form, the expression of c_3 in (35) is actually symmetric in M and N. The term $\pi M/24N$ in c_3 now yields the central charge $c = 1$ upon taken $M = N$. This agrees with the accepted value for dimer and Ising systems.

For an $M \times N$ dimer lattice with one vacant boundary site and both $M, N =$ odd, we use (31) for $N_{\text{dimer}}(\mathcal{L}_D)$ and the conversion of $M = 2\mathcal{M} - 1, N = 2\mathcal{N} - 1$ to get the desired result. After some algebra, one obtains the same $f_{\text{bulk}}, c_1, c_2$ as in (35), and a new c_3 given by

$$c_3' = -\frac{1}{2}\ln N + \frac{G}{\pi} + \frac{7}{4}\ln 2 - \ln(1+\sqrt{2}) - \frac{\pi M}{12N} + \sum_{n=1}^{\infty}\ln\left(1 - e^{-2n\pi M/N}\right). \tag{36}$$

The expression of c_3', which is again symmetric in $\mathcal{M}, \mathcal{N}$, leads to a new central charge $c = -2$. Furthermore, the term $-\frac{1}{2}\ln N$ in c_3', which is absent in c_3, gives a $\sqrt{N}$ correction to the dimer enumeration. A concrete example exhibiting this correction has been given in Ref. 8. We remark that Kenyon[18] has found the correction factor to be $N^{3/4}$ if the vacancy occurs in the interior of the lattice.

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HYPERBOLIC STRUCTURE ARISING FROM A KNOT INVARIANT II: COMPLETENESS

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We study the geometry of a knot invariant defined in terms of the quantum dilogarithm function. We show that a hyperbolic structure naturally arises in the classical limit of the invariant; the completeness conditions can also be identified with the saddle point equations by studying a (1,1)-tangle.

1. Introduction

After the discovery of quantum groups, the interest in quantum invariants has been renewed, and we have now infinitely many knot invariants. In contrast to the fact that the Alexander invariant was defined from the homology of the universal Abelian covering, the geometrical meaning of those quantum invariants remains unclear. A key to solving these problems comes from Kashaev's observation[1,2] that the asymptotic behavior of Kashaev's knot invariant, which was later shown[3] to coincide with a specific value of the colored Jones polynomial, gives a hyperbolic volume of the knot complement,

$$\|S^3 \setminus K\| = \frac{1}{v_3} \lim_{N\to\infty} \frac{2\pi}{N} \log |J_N(K)|,$$

where $\|\cdot\|$ is the Gromov norm, v_3 is the hyperbolic volume of the regular ideal tetrahedron, and $J_N(K) = V_N(K; e^{2\pi i/N})$ is given in terms of the colored Jones polynomial $V_N(K;t)$ for the knot K (N-dimensional representation of $s\ell_2$).

This paper is a continuation of Ref. 4. Therein we showed that a hyperbolic structure naturally arises from a knot invariant, which was defined by use of the infinite-dimensional representation of the quantum dilogarithm function. In this sense this invariant can be viewed as a non-compact analogue of Kashaev's invariant. We found that the saddle point equations which denote a critical point of our invariant, coincide with the hyperbolicity consistency conditions in gluing ideal polyhedra. One purpose of this paper is to prove that the completeness condition is also given through a classical limit of our knot invariant $\tau_1(K)$ together with a suitable (1,1)-tangle. See Ref. 5 for a geometrical study of Kashaev's original invariant.

This paper is organized as follows. We briefly review the construction of our knot invariant in Sec. 2. The essential tool is the S-operator which solves the five-term relation. In Sec. 3 we give a 3-dimensional picture[4] of our invariant by assigning an oriented tetrahedron to the S-operator. In the classical limit the tetrahedron can be regarded as an ideal hyperbolic tetrahedron. We clarify how completeness is given from our invariant by studying a related (1,1)-tangle. In Sec. 4 we define a quantum invariant of 3-manifold M based on the ideal triangulation of M. The last section is devoted to concluding remarks. We discuss the relationship between the classical limit of the quantum invariant and the hyperbolic volume. Throughout this paper we use Euler's dilogarithm $\mathrm{Li}_2(z)$, Rogers' dilogarithm $L(z)$, and the Bloch–Wigner function $D(z)$, which are respectively defined by

$$\mathrm{Li}_2(z) = -\int_0^z \mathrm{d}s \frac{\log(1-s)}{s}, \qquad L(z) = \mathrm{Li}_2(z) + \frac{1}{2}\log(z)\log(1-z),$$
$$D(z) = \mathrm{Im}\mathrm{Li}_2(z) + \arg(1-z)\log|z|.$$

2. Knot Invariant

We introduce the S-operator, acting on two spaces $\mathbf{V} \otimes \mathbf{V}$,

$$S_{1,2} = e^{\frac{1}{2\mathrm{i}\gamma}\hat{q}_1\hat{p}_2}\,\Phi_\gamma(\hat{p}_1 + \hat{q}_2 - \hat{p}_2). \tag{1}$$

Here we have used the canonical operators, $[\hat{p}_j, \hat{q}_k] = -2\mathrm{i}\gamma\delta_{j,k}$, and

$$\Phi_\gamma(\varphi) = \exp\left(\int_{\mathbb{R}+\mathrm{i}0} \frac{e^{-\mathrm{i}\varphi x}}{4\sinh(\gamma x)\sinh(\pi x)}\frac{\mathrm{d}x}{x}\right), \tag{2}$$

which can be regarded as a modular double of the q-exponential function.[6] The S-operator satisfies the five-term relation,[7–9]

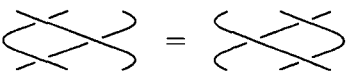
$$S_{2,3}S_{1,2} = S_{1,2}S_{1,3}S_{2,3}. \tag{3}$$

By recursive use of this five-term relation, we see that the R-operator

$$R_{12,34} = (S_{1,4}^{t_4})^{-1}S_{1,3}S_{2,4}^{t_2t_4}(S_{2,3}^{t_2})^{-1} \equiv P_{1,3}P_{2,4}\check{R}_{12,34}, \tag{4}$$

where t_a is the transposition on the a-th space and $P_{a,b}$ is the permutation operator, solves the Yang–Baxter equation (braid relation),

$$\check{R}_{12,34}\check{R}_{34,56}\check{R}_{12,34} = \check{R}_{34,56}\check{R}_{12,34}\check{R}_{34,56}; \tag{5}$$



We define a knot invariant from this R-matrix. For this purpose we give the matrix elements and their classical limit ($\gamma \to 0$) in the case of $\mathbf{V}$ being the momentum space, $\hat{p}|p\rangle = p|p\rangle$ with $p \in \mathbb{R}$;

$$\langle p_1, p_2|S_{1,2}|p_1', p_2'\rangle = \delta(p_1 + p_2 - p_1')\Phi_\gamma(p_2' - p_2 + \mathrm{i}\pi + \mathrm{i}\gamma)e^{\frac{1}{2\mathrm{i}\gamma}(-\frac{\pi^2+\gamma^2}{6} - \frac{\gamma\pi}{2} + p_1(p_2'-p_2))}$$
$$\sim \delta(p_1 + p_2 - p_1')\exp\left(-\frac{1}{2\,\mathrm{i}\,\gamma}V(p_2' - p_2, p_1)\right), \tag{6}$$

$$\langle p_1, p_2 | S_{1,2}^{-1} | p_1', p_2' \rangle = \delta(p_1 - p_1' - p_2') \frac{1}{\Phi_\gamma(p_2 - p_2' - \mathrm{i}\pi - \mathrm{i}\gamma)} e^{\frac{1}{2\mathrm{i}\gamma}(\frac{\pi^2+\gamma^2}{6} + \frac{\gamma\pi}{2} - p_1'(p_2 - p_2'))}$$
$$\sim \delta(p_1 - p_1' - p_2') \exp\left(\frac{1}{2\,\mathrm{i}\,\gamma} V(p_2 - p_2', p_1')\right). \tag{7}$$

Here we have set

$$V(x,y) = \frac{\pi^2}{6} - \mathrm{Li}_2(e^x) - xy, \tag{8}$$

which satisfies

$$V(x,y) = L(1-e^x) + \frac{1}{2}\left(x\frac{\partial V(x,y)}{\partial x} + y\frac{\partial V(x,y)}{\partial y}\right), \tag{9}$$
$$\mathrm{Im}V(x,y) = D(1-e^x) + \log|e^x| \cdot \mathrm{Im}\left(\frac{\partial V(x,y)}{\partial x}\right) + \log|e^y| \cdot \mathrm{Im}\left(\frac{\partial V(x,y)}{\partial y}\right).$$

Recalling the fact that the volume of an ideal tetrahedron in 3-dimensional hyperbolic space $\mathbb{H}^3$ is written in terms of the Bloch–Wigner function,[10,11] and that the Rogers dilogarithm function is a natural complexification of the hyperbolic volume,[12,13] we can expect that the S-operator at the critical point is closely connected with an ideal tetrahedron in $\mathbb{H}^3$.

We can define the knot invariant from the enhanced Yang–Baxter operator[14] $(\check{R}, \mu, \alpha, \beta)$,

$$\tau_1(L) = \alpha^{-w(\xi)}\beta^{-n}\,\mathrm{Tr}_{2,\ldots,n}\left(b_{\check{R}}(\xi)\left(1 \otimes \mu^{\otimes(n-1)}\right)\right). \tag{10}$$

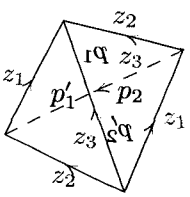
Here ξ denotes the braid group representation with n strands of link L, and $b_{\check{R}}(\xi)$ is to substitute the operator $\check{R}$ (4) as a braid generator. We have used $w(\xi)$ as writhe, and

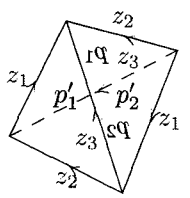
$$\mu = e^{\frac{\pi+\gamma}{\gamma}\hat{p}} \otimes e^{-\frac{\pi+\gamma}{\gamma}\hat{p}^t}, \quad \alpha = e^{-\mathrm{i}\frac{\pi^2+\gamma^2}{\gamma}}, \quad \beta = \frac{\gamma\, e^{\mathrm{i}\frac{\pi^2+\gamma^2}{\gamma}}}{(1-e^{2\mathrm{i}\gamma})(1-e^{2\mathrm{i}\pi^2/\gamma})}.$$

Note that $\tau_1(K)$ is an invariant for (1,1)-tangles, and that we do not take trace over the first space.

3. Hyperbolicity and Completeness

We have clarified in Ref. 4 how the hyperbolic structure naturally arises from the knot invariant $\tau_1(K)$ in the classical limit $\gamma \to 0$. Therein, motivated by the fact that the S-operator solves the five-term relation (3), the S-operator is identified in the classical limit with an ideal tetrahedron in $\mathbb{H}^3$,

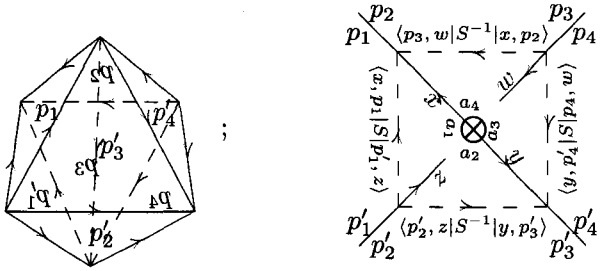
$\langle p_1, p_2 | S | p_1', p_2' \rangle =$ 

$\langle p_1, p_2 | S^{-1} | p_1', p_2' \rangle =$ 

(11)

Here momenta p_a and p'_a are assigned to every face. Both oriented ideal tetrahedra have modulus $z = e^{p'_2 - p_2}$, and every dihedral angle is fixed as a function of the modulus, $z_1(z) = z$, $z_2(z) = 1 - z^{-1}$, and $z_3(z) = (1-z)^{-1}$, opposite edges having the same angles. The integral with respect to momentum p_a is interpreted as gluing two faces having the same momentum to match the orientation of every edge. One sees that the five-term relation (3) is simply realized as the $2 \leftrightarrow 3$ Pachner move. Correspondingly, the R-matrix (4) can be depicted as an oriented ideal octahedron,

$$\langle \vec{p}|\check{R}|\vec{p'}\rangle = \quad ; \tag{12}$$



Here we have $p_1 + p_3 = p'_1 + p_2$ and $p'_2 + p'_4 = p'_3 + p_4$. The picture on the right denotes the projection of the octahedron viewed from the top of the octahedron, and a_i is the dihedral angle around a central axis "$\otimes$," $a_1 = z_3(e^{z-p_1})$, $a_2 = z_2(e^{p'_3 - z})$, $a_3 = z_3(e^{w-p'_4})$, and $a_4 = z_2(e^{p_2-w})$, satisfying the consistency condition,

$$\prod_{i=1}^{4} a_i = 1; \quad e^w = \frac{1 - e^{p_1 - p'_1 + p_4 - p'_4}}{e^{-p'_4} - e^{p_1 - p'_1 + p_4 - p'_4 - p_2}}, \quad e^z = \frac{1 - e^{p_1 - p'_1 + p_4 - p'_4}}{e^{-p'_3} - e^{p_1 - p'_1 + p_4 - p'_4 - p_1}}. \tag{13}$$

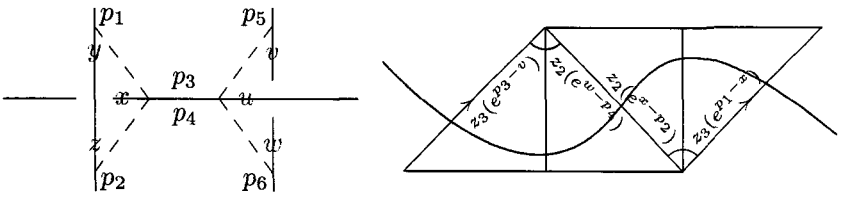
Note that we have symmetry of the R-matrix, and that the inverse braiding generator $\check{R}^{-1}$ is also given by an oriented ideal octahedron,

$$\langle p_1, p_2, p_3, p_4|\check{R}|p'_1, p'_2, p'_3, p'_4\rangle = \langle p'_4, p'_3, p'_2, p'_1|\check{R}|p_4, p_3, p_2, p_1\rangle,$$

$$\langle p_1, p_2, p_3, p_4|\check{R}^{-1}|p'_1, p'_2, p'_3, p'_4\rangle = \langle p'_2, p'_1, p_1, p_2|\check{R}|p'_3, p'_4, p_4, p_3\rangle.$$

Our main claim in Ref. 4 is that the saddle point conditions which are derived from the classical limit of $\tau_1(K)$, exactly coincide with the hyperbolicity consistency conditions in gluing ideal tetrahedra. Generally, to endow the hyperbolic structure in a 3-manifold M which is constructed by gluing together a finite collection of ideal tetrahedra, we should check the completeness condition besides the hyperbolicity consistency conditions.[10] We shall show that this completeness condition is fulfilled by considering the knot invariant as coming from a constituent (1,1)-tangle.

In computing $\tau_1(L)$ from a (1,1)-tangle, we cut the link L at a point which is located on an alternating segment of L (left figure below),



We glue together faces, p_3 and p_4, and part of the developing map is written as the right figure above (see Ref. 5 for some discussion). The completeness condition can be read from this picture as

$$\frac{1-e^{p_4-w}}{1-e^{p_3-v}} = \frac{1-e^{p_2-x}}{1-e^{p_1-x}}. \tag{14}$$

On the other hand, the contribution from this segment to the invariant is given by

$$\int \mathrm{d}x\mathrm{d}u \, \langle p_3, x|S^{-1}|y, p_1\rangle \, \langle p_5, v|S^{-1}|u, p_3\rangle \, \langle z, p_2|S|p_4, x\rangle \, \langle u, p_4|S|p_6, w\rangle\big|_{p_3=p_4=P}.$$

Here we have assumed $p_3 = p_4 = P$ as we are studying the (1,1)-tangle. Substituting Eqs. (6)–(7) as the limit $\gamma \to 0$, we obtain the saddle point condition as

$$\frac{1-e^{p_1-x}}{1-e^{p_2-x}} = 1. \tag{15}$$

When we set $p_3 = p_4 = P = \pm\infty$, we find that the saddle point equation (15) coincides with the completeness condition (14). We can see that other completeness conditions can be deduced from Eq. (14) with the help of the hyperbolic consistency conditions (13) in constructing the R-matrix. As a result, by constructing the invariant of link L from a constituent (1,1)-tangle and substituting a specific value therein, we can see a correspondence between the completeness conditions and saddle point equations. Combining our previous result[4] that the hyperbolicity consistency conditions coincide with the saddle point equations, we can conclude that the invariant $\tau_1(L)$ with the 3-dimensional picture (11) gives an ideal triangulation of the knot complement, and that Eq.(9) will indicate a coincidence between the asymptotic value of our invariant at the critical point and the hyperbolic volume of the knot complement.

4. Quantum Invariant of Manifold

We define a quantum invariant of manifold M in terms of the S-operator (1). When 3-manifold M admits a hyperbolic structure of finite volume, the 3-manifold M is constructed by gluing a finite collection of oriented ideal tetrahedra. We assign an S-operator to each oriented ideal tetrahedron, and we define the partition function of M by[15]

$$Z(M) = \iint \mathrm{d}p \prod \langle p_i, p_j|S^{\pm 1}|p_k, p_l\rangle. \tag{16}$$

Here we need to assume a constraint for p which corresponds to the completeness condition of M. As was studied in the previous section for the case of the knot invariant $\tau_1(L)$, the hyperbolicity consistency conditions are given as the saddle point equations in the integral over p, while the completeness conditions are controlled by considering a (1,1)-tangle of link L.

It should be noted that the idea of assigning a solution of the five-term relation to a tetrahedron,[16,17] where the Regge action is derived from the asymptotic value

of the classical $6j$-symbol for large j, is well known. We have rather clarified that the S-operator (1) denotes a quantization of the hyperbolic ideal tetrahedron.

4.1. *Examples*

We consider knots $K = 4_1$ and 5_2,

As is well-known, these knots are hyperbolic, and the complements of the knots can be constructed by gluing ideal tetrahedra (see, *e.g.* Refs. 18, 19). Indeed the ideal triangulation can be done following Ref. 4, and we get the quantum invariant by assigning S-operators to each oriented ideal tetrahedron,

$$Z(S^3 \setminus 4_1) = \int \mathrm{d}p\, \delta_{p_4-p_2=p_1-p_3}\, \langle p_1, p_2|S|p_3, p_4\rangle\, \langle p_4, p_3|S^{-1}|p_2, p_1\rangle,$$

$$Z(S^3 \setminus 5_2) = \int \mathrm{d}p\, \delta_{p_5-p_4=p_2-p_6}\, \langle p_1, p_5|S^{-1}|p_4, p_3\rangle \times \langle p_2, p_4|S^{-1}|p_6, p_5\rangle\, \langle p_3, p_6|S^{-1}|p_1, p_2\rangle.$$

The constraint $\delta...$ represents the completeness condition. In the small-γ limit, these integrals respectively reduce to

$$Z(S^3 \setminus 4_1) \sim \int \mathrm{d}x \exp \frac{1}{2\,\mathrm{i}\,\gamma} \left(\mathrm{Li}_2(e^{-x}) - \mathrm{Li}_2(e^{x})\right),$$

$$Z(S^3 \setminus 5_2) \sim \iint \mathrm{d}x\mathrm{d}y \exp \frac{1}{2\,\mathrm{i}\,\gamma} \Big(\frac{\pi^2}{2} - \mathrm{Li}_2(e^{y-x}) - 2\,\mathrm{Li}_2(e^{-y}) - y(y-x)\Big).$$

After applying the saddle point method whose conditions exactly coincide with the hyperbolicity gluing conditions and completeness conditions, we see

$$2\gamma \log Z(S^3 \setminus K) \sim \begin{cases} 2.029883212819307, & \text{for } K = 4_1 \\ 2.828122088330783 + 3.024128376509301\,\mathrm{i}, & \text{for } K = 5_2 \end{cases}$$

Comparing with a table in Ref. 20 (it is necessary to multiply the Chern–Simons terms there by $2\pi^2$), this result suggests the "VCS conjecture",[3,5,15,21]

$$2\gamma \log Z(M) \sim \mathrm{Vol}(M) + \mathrm{i}\,\mathrm{CS}(M) \equiv \mathrm{VCS}(M), \tag{17}$$

where Vol and CS respectively denote the hyperbolic volume and the Chern–Simons invariant of M.

5. Concluding Remarks

We have studied how the knot invariant $\tau_1(L)$ is related with the hyperbolic geometry $\mathbb{H}^3$ in the limit $\gamma \to 0$. We have shown that not only the hyperbolicity consistency conditions but also the completeness conditions can be derived from

the saddle point equations when we consider the invariant of a link L as resulting from the invariant of a (1,1)-tangle.

Based on the S-operator (1), we have defined the partition function (16) of 3-manifold M. Once M is triangulated into a finite collection of ideal tetrahedra, the volume $\mathrm{Vol}(M)$ is given by a summation, $\sum_i D(z_i)$, where modulus z_i satisfies a set of hyperbolicity and complete conditions. Recalling that the S-operator and its imaginary part respectively reduce to the Rogers dilogarithm function $L(z)$ and the Bloch–Wigner function $D(z)$ at the critical point (9), the conjecture (17) seems to be reasonable. However, the Rogers dilogarithm function $L(z)$ is a multi-valued function of z, with singularities at 0 and 1, and the value on the universal abelian cover of $\mathbb{C}\setminus\{0,1\}$ is given with an integer pair (c_1, c_2) as[13] $L(z) + \frac{i\pi}{2}(c_1 \log(1-z) + c_2 \log(z))$. In computations such as for $K = 6_1$, we need such terms to get "correct" answers. It remains for future studies to discover how to specify the branch of $L(z)$ in the small-γ limit.

From a physical point of view, the hyperbolic geometry or the Euclidean AdS receives much attention based on the AdS/CFT correspondence. As it is well known that the Einstein–Hilbert action can be rewritten in terms of the CS action,[22,23] our result which relates the partition function $Z(M)$ with $\mathbb{H}^3$ and CS will be promising in further studies of quantum gravity.

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EXOTIC GALILEAN SYMMETRY AND THE HALL EFFECT *

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The "Laughlin" picture of the Fractional Quantum Hall effect can be derived using the "exotic" model based on the two-fold centrally-extended planar Galilei group. When coupled to a planar magnetic field of critical strength determined by the extension parameters, the system becomes singular, and "Faddeev-Jackiw" reduction yields the "Chern-Simons" mechanics of Dunne, Jackiw, and Trugenberger. The reduced system moves according to the Hall law.

1. Introduction

In his seminal paper[1] Laughlin argued that the Fractional Quantum Hall Effect[2] could be explained as condensation into a collective ground state, represented by the lowest-Landau-level wave functions

$$f(z)e^{-|z|^2/4}, \tag{1}$$

where the complex N-vector z denotes the positions of N polarized electrons in the plane; $f(z)$ is analytic. The fundamental operators are $\widehat{z}f = zf$, and $\widehat{\bar{z}}f = 2\partial_z f$ satisfy $\left[\widehat{\bar{z}}, \widehat{z}\right] = 2$. The quantum Hamiltonian only involves the potential $V(\bar{z}, z)$ suitably quantized with the choice of an ordering for the non-commuting operators $\widehat{\bar{z}}$ and $\widehat{z}$.

Our results[3] presented here say that the Laughlin picture can actually be obtained from *first principles*, namely using the two-fold central extension of the planar Galilei group. This latter has been known for some time,[4,5] but has long remained a kind of curiosity, since it had no obvious physical use: for a free particle of mass m, the extra structure related to the new invariant k leaves the usual motions unchanged, and only contributes to the conserved quantities.[3,5,6] Let us

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mention that our "exotic" theory is in fact equivalent to Quantum Mechanics in the non-commutative plane,[7] with non-commutative parameter $\theta = k/m^2$.

Coupling an "exotic" particle to an electromagnetic field, the two extension parameters, k and m, combine with the magnetic field, B, into an effective mass, m^*, given by (4); when this latter vanishes, the consistency of the equations of motion requires that the particle obey the Hall law. Interestingly, for $m^* = 0$, Hamiltonian reduction[8] yields the "Chern-Simons mechanics" considered before by Dunne, Jackiw and Trugenberger.[9] The reduced theory admits the infinite symmetry of area-preserving diffeomorphisms, found before for the edge currents of the Quantum Hall states.[10]

2. Exotic particle in a gauge field

Let us consider the action

$$\int (\vec{p} - \vec{A}) \cdot d\vec{x} - h\, dt + \frac{\theta}{2}\, \vec{p} \times d\vec{p}, \tag{2}$$

where $(V, \vec{A})$ is an electro-magnetic potential, the Hamiltonian being $h = \vec{p}^{\,2}/2m + V$. The term proportional to the non-commutative parameter θ is actually equivalent to the acceleration-dependent Lagrangian of Lukierski et al.[6]The associated Euler-Lagrange equations read

$$\begin{cases} m^* \dot{x}_i = p_i - m\theta\, \varepsilon_{ij} E_j, \\ \dot{p}_i = E_i + B\, \varepsilon_{ij} \dot{x}_j, \end{cases} \tag{3}$$

where we have introduced the *effective mass*

$$m^* = m(1 - \theta B). \tag{4}$$

The velocity and momentum are different if $\theta \neq 0$. The equations of motions (3) can also be written as

$$\omega_{\alpha\beta} \dot{\xi}_\beta = \frac{\partial h}{\partial \xi_\alpha}, \qquad \text{where} \qquad (\omega_{\alpha\beta}) = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & B \\ 0 & -1 & -B & 0 \end{pmatrix}. \tag{5}$$

Note that the electric and magnetic fields are otherwise arbitrary solutions of the homogeneous Maxwell equation $\partial_t B + \varepsilon_{ij} \partial_i E_j = 0$, which guarantees that the two-form $\omega = \frac{1}{2} \omega_{\alpha\beta} d\xi_\alpha \wedge d\xi_\beta$ is closed, $d\omega = 0$.

When $m^* \neq 0$, the determinant $\det(\omega_{\alpha\beta}) = (1-\theta B)^2 = (m^*/m)^2$ is nonzero and the matrix $(\omega_{\alpha\beta})$ in (5) can be inverted. Then the equations of motion (5) (or (3)) take the form $\dot{\xi}_\alpha = \{\xi_\alpha, h\}$, with the standard Hamiltonian, but with the new Poisson bracket $\{f,g\} = (\omega^{-1})_{\alpha\beta}\partial_\alpha f \partial_\beta g$ which reads, explicitly,

$$\begin{aligned}\{f,g\} = \frac{m}{m^*}\Bigg[&\frac{\partial f}{\partial \vec{x}}\cdot\frac{\partial g}{\partial \vec{p}} - \frac{\partial g}{\partial \vec{x}}\cdot\frac{\partial f}{\partial \vec{p}} \\ &+ \theta\left(\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_2}\right) + B\left(\frac{\partial f}{\partial p_1}\frac{\partial g}{\partial p_2} - \frac{\partial g}{\partial p_1}\frac{\partial f}{\partial p_2}\right)\Bigg].\end{aligned} \tag{6}$$

Further insight can be gained when the magnetic field B is a (positive) nonzero constant, which turns out the most interesting case, and will be henceforth assumed. (The electric field $E_i = -\partial_i V$ is still arbitrary). Introducing the new coordinates

$$\begin{cases} Q_i = x_i + \dfrac{1}{B}\left[1 - \sqrt{\dfrac{m^*}{m}}\right]\varepsilon_{ij}\, p_j, \\ P_i = \sqrt{\dfrac{m^*}{m}}\, p_i - \dfrac{1}{2}B\varepsilon_{ij}\, Q_j, \end{cases} \tag{7}$$

will allow us to generalize our results in Ref. 3 from a constant to any electric field.

Firstly, the Cartan one-form[12] in the action (2) reads simply $P_i dQ_i - h dt$, so that the symplectic form on phase space retains the canonical guise, $\omega = dP_i \wedge dQ_i$. The price to pay is that the Hamiltonian becomes rather complicated.[3]

The equations of motion (3) are conveniently presented in terms of the new variables $\vec{Q}$ and the old momenta $\vec{p}$, as

$$\begin{cases} \dot{Q}_i = \varepsilon_{ij}\dfrac{E_j}{B} + \sqrt{\dfrac{m}{m^*}}\left(\dfrac{p_i}{m} - \varepsilon_{ij}\dfrac{E_j}{B}\right), \\ \dot{p}_i = \varepsilon_{ij}B\dfrac{m}{m^*}\left(\dfrac{p_j}{m} - \varepsilon_{jk}\dfrac{E_k}{B}\right). \end{cases} \tag{8}$$

Note that all these expressions diverge when m^* tends to zero.

When the magnetic field takes the particular value

$$B = B_c = \frac{1}{\theta}, \tag{9}$$

the effective mass (4) vanishes, $m^* = 0$, so that $\det(\omega_{\alpha\beta}) = 0$, and the system becomes singular. Then the time derivatives $\dot{\xi}_\alpha$ can no longer be expressed from the variational equations (5), and we have resort to "Faddeev-Jackiw" reduction.[8] In accordance with the Darboux theorem (see, e.g., Ref. 12), the Cartan one-form in (2) can be written, up to an exact term, as

$$\vartheta - h dt, \qquad \text{with} \qquad \vartheta = (p_i - \frac{1}{2}B_c\,\varepsilon_{ij}\,x_j)dx_i + \frac{1}{2}\theta\varepsilon_{ij}\,p_i dp_j = P_i dQ_i, \tag{10}$$

where the new coordinates read, consistently with (7),

$$Q_i = x_i + \frac{1}{B_c}\varepsilon_{ij}p_j, \tag{11}$$

while the $P_i = -\frac{1}{2}B_c\,\varepsilon_{ij}\,Q_j$ are in fact the rotated coordinates Q_i. Eliminating the original coordinates $\vec{x}$ and $\vec{p}$ using (11), we see that the Cartan one-form reads $P_i dQ_i - H(\vec{Q},\vec{p})dt$, where $H(\vec{Q},\vec{p}) = \vec{p}^{\,2}/(2m) + V(\vec{Q},\vec{p})$. As the p_i appear here with no derivatives, they can be eliminated using their equation of motion $\partial H(\vec{Q},\vec{p})/\partial\vec{p} = 0$, i. e., the constraint

$$\frac{p_i}{m} - \frac{\varepsilon_{ij}E_j}{B_c} = 0. \tag{12}$$

A short calculation shows that the reduced Hamiltonian is just the original potential, viewed as a function of the "twisted" coordinates $\vec{Q}$, *viz.*

$$H = V(\vec{Q}). \tag{13}$$

This rule is referred to as the "Peierls substitution".[3,9] Since $\partial^2 H/\partial p_i \partial p_j = \delta_{ij}/m$ is already non singular, the reduction stops, and we end up with the reduced Lagrangian

$$L_{\text{red}} = \frac{1}{2\theta}\vec{Q}\times\dot{\vec{Q}} - V(\vec{Q}), \tag{14}$$

supplemented with the Hall constraint (12). The 4-dimensional phase space is hence reduced to 2 dimensions, with Q_1 and Q_2 in (11) as canonical coordinates, and reduced symplectic two-form $\omega_{\text{red}} = \frac{1}{2}B_c\,\varepsilon_{ij}dQ_i\wedge dQ_j$ so that the reduced Poisson bracket is

$$\{F,G\}_{\text{red}} = -\frac{1}{B_c}\Big(\frac{\partial F}{\partial Q_1}\frac{\partial G}{\partial Q_2} - \frac{\partial G}{\partial Q_1}\frac{\partial F}{\partial Q_2}\Big). \tag{15}$$

The twisted coordinates are therefore again non-commuting,

$$\{Q_1,Q_2\}_{\text{red}} = -\theta = -\frac{1}{B_c}. \tag{16}$$

The equations of motion associated with (14), and also consistent with the Hamilton equations $\dot{Q}_i = \{Q_i,H\}_{\text{red}}$, are given by

$$\dot{Q}_i = \varepsilon_{ij}\frac{E_j}{B_c}, \tag{17}$$

in accordance with the Hall law (compare (8) with the divergent terms removed).

Putting $B_c = 1/\theta$, the Lagrangian (14) becomes formally identical to the one Dunne et al.[9] derived letting the *real* mass go to zero. Note, however, that while $\vec{Q}$ denotes real position in Ref. 9, our $\vec{Q}$ here is the "twisted" expression (11), with the magnetic field frozen at the critical value $B_c = 1/\theta$.

3. Infinite symmetry

It has been argued[11] that the physical process which yields the Fractional Quantum Hall Effect actually takes place at the boundary of the droplet of the "Hall" liquid: owing to incompressibility, the bulk can not support any density waves, but there are chiral currents at the edge. These latter fall into irreducible representations

of the infinite dimensional algebra $W_{1+\infty}$,[10] which is the quantum deformation of w_∞, the algebra of classical observables which generate the group of area-preserving diffeomorphisms of the plane.

Our reduced model is readily seen to admit w_∞, the classical counterpart of $W_{1+\infty}$, as symmetry. To see this, let us remember that, as argued by Souriau,[12] and later by Crnkovic and Witten,[13] it is convenient to consider the space of solutions of the equations of motion (Souriau's "*espace des mouvements*" [= space of motions]), denoted by $\mathcal{M}$. For a classical mechanical system, this is an abstract substitute for the classical phase space, whose points are the motion curves of the system. The classical dynamics is encoded into the symplectic form Ω of $\mathcal{M}$. It is then obvious that *any* function $f(\zeta)$ on $\mathcal{M}$ is a constant of the motion. (When expressed using the positions, time, and momenta, such a function can look rather complicated). Any such function $f(\zeta)$ generates a Hamiltonian vectorfield Z^μ on $\mathcal{M}$ through the relation

$$-\partial_\mu f = \Omega_{\mu\nu} Z^\nu. \tag{18}$$

The vector field Z^μ generates, at least locally, a 1-parameter group of diffeomorphisms of $\mathcal{M}$. All diffeomorphisms of $\mathcal{M}$ which leave the symplectic form Ω invariant form an infinite dimensional group, namely the group of symplectomorphisms of $\mathcal{M}$. Any symplectic transformation is a symmetry of the system : it merely permutes the motions curves.

For the reduced system above, the reduced phase space is two dimensional. The space of motions is therefore locally a plane. (Its global structure plainly depends on the details of the dynamics). Now, for any orientable two dimensional manifold, the symplectic form is the area element; it follows that the reduced system admits the group of area-preserving transformations as symmetry.

4. Quantization

Let us conclude our general theory by quantizing the coupled system. Again, owing to the exotic term, the position representation does not exist.

Introducing the complex coordinates

$$\begin{cases} z = \dfrac{\sqrt{B}}{2}(Q_1 + iQ_2) + \dfrac{1}{\sqrt{B}}(-iP_1 + P_2) \\ w = \dfrac{\sqrt{B}}{2}(Q_1 - iQ_2) + \dfrac{1}{\sqrt{B}}(-iP_1 - P_2) \end{cases} \tag{19}$$

the two-form $dP_i \wedge dQ_i$ on 4-dimensional unreduced phase space becomes the canonical Kähler two-form of $\mathbf{C}^2$, viz $\omega = (2i)^{-1}(d\bar{z} \wedge dz + d\bar{w} \wedge dw)$. choosing the antiholomorphic polarization, the "unreduced" quantum Hilbert space, consisting of the "Bargmann-Fock" wave functions

$$\psi(z, \bar{z}, w, \bar{w}) = f(z, w) e^{-\frac{1}{4}(z\bar{z} + w\bar{w})}, \tag{20}$$

where f is holomorphic in both of its variables. The fundamental quantum operators,

$$\begin{cases} \widehat{z}\, f = zf, \ \widehat{\bar{z}}\, f = 2\partial_z f, \\ \widehat{w} f = wf, \widehat{\bar{w}} f = 2\partial_w f, \end{cases} \tag{21}$$

satisfy the commutation relations $\left[\widehat{\bar{z}}, \widehat{z}\right] = \left[\widehat{\bar{w}}, \widehat{w}\right] = 2$, and $\left[\widehat{z}, \widehat{w}\right] = \left[\widehat{\bar{z}}, \widehat{\bar{w}}\right] = 0$. We recognize here the familiar creation and annihilation operators, namely $a_z^* = z$, $a_w^* = w$, and $a_z = \partial_z$, $a_w = \partial_w$. Using (7), the (complex) momentum $p = p_1 + ip_2$ and the kinetic part, h_0, of the Hamiltonian become, respectively,

$$p = -i\sqrt{\frac{mB}{m^*}}\,\bar{w} \qquad \text{and} \qquad h_0 = \frac{B}{2m^*}\, w\bar{w}. \tag{22}$$

For $m^* \neq 0$ the wave function satisfies the Schrödinger equation $i\partial_t f = \widehat{h} f$, with $\widehat{h} = \widehat{h}_0 + \widehat{V}$. The quadratic kinetic term here is

$$\widehat{h}_0 = \frac{B}{4m^*}\left(\widehat{w}\widehat{\bar{w}} + \widehat{\bar{w}}\,\widehat{w}\right) = \frac{B}{2m^*}\left(\widehat{w}\widehat{\bar{w}} + 1\right). \tag{23}$$

The case when the effective mass tends to zero is conveniently studied in this framework. On the one hand, in the limit $m^* \to 0$, one has

$$z \to \sqrt{B}\, Q, \qquad w \to 0, \tag{24}$$

where $Q = Q_1 + iQ_2$, cf. (7); the 4-dimensional phase space reduces to the complex plane. On the other hand, from (22) and (21) we deduce that

$$i\sqrt{\frac{m^*}{mB}}\,\widehat{p} = \widehat{\bar{w}} = 2\partial_w. \tag{25}$$

The limit $m^* \to 0$ is hence enforced, at the quantum level, by requiring that the wave functions be independent of the coordinate w, i.e.,

$$\partial_w f = 0, \tag{26}$$

yielding the reduced wave functions of the form

$$\Psi(z, \bar{z}) = f(z) e^{-\frac{1}{4} z\bar{z}}, \tag{27}$$

where f is a holomorphic function of the reduced phase space parametrized by z. When viewed in the "big" Hilbert space (see (20)), these wave functions belong, by (23), to the lowest Landau level.[2,3,14]

Using the fundamental operators $\widehat{z}$ an $\widehat{\bar{z}}$ given in (21), we easily see that the (complex) "physical" position $x = x_1 + ix_2$ and its quantum counterpart $\widehat{x}$, namely

$$x = \frac{1}{\sqrt{B_c}}\left(z + \sqrt{\frac{m}{m^*}}\,\bar{w}\right), \qquad \widehat{x} = \frac{1}{\sqrt{B_c}}\left(z + \sqrt{\frac{m}{m^*}}\, 2\partial_w\right), \tag{28}$$

manifestly diverge when $m^* \to 0$. *Positing from the outset* the conditions (26) the divergence is suppressed, however, leaving us with the reduced position operators

$$\widehat{x}\, f = \widehat{Q} f = \frac{1}{\sqrt{B_c}}\, zf, \qquad \widehat{\bar{x}}\, f = \widehat{\bar{Q}} f = \frac{2}{\sqrt{B_c}}\, \partial_z f, \tag{29}$$

whose commutator is $[\widehat{Q},\widehat{\bar{Q}}] = 2/B_c$, cf. (16). In conclusion, we recover the "Laughlin" description (1) of the ground states of the FQHE in Ref. 2. Quantization of the reduced Hamiltonian (which is, indeed, the potential $V(z,\bar{z})$), can be achieved using, for instance, anti-normal ordering.[9,14]

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THE THREE-STATE CHIRAL CLOCK MODEL*

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We give a brief summary of our recent works on the three-state chiral clock model. In these works, we use improved effective field theories with clusters being strips, infinite in the chiral direction and finite in the non-chiral direction. Hence, effective-field transfer matrix methods can be employed in these studies. The effective fields are determined by the Gibbs-Bogoliubov free energy variational principle, leading to Weiss or Bethe approximations in different studies respectively. By systematic improvement of these approximations, i.e. widening the strips, these studies point to the conclusion that there is no Lifshitz point existing at finite non-zero chirality.

1. Introduction

The three-state chiral clock model was introduced independently by Ostlund[1] and Huse.[2] It is the simplest model with only nearest-neighbor interactions which exhibits spatially modulated phases. These spatially modulated phases occur diversely in physical systems.[3] The reduced Hamiltonian for this model on the two-dimensional square lattice is

$$-\beta H(\{n_{i,j}\},\Delta)=\sum_{i,j}\left[K_n\cos\frac{2\pi}{3}(n_{i,j}-n_{i,j+1}+\Delta)+K_t\cos\frac{2\pi}{3}(n_{i,j}-n_{i+1,j})\right], \tag{1}$$

where $\beta=1/k_{\mathrm{B}}T$. From the symmetry within this model, we can restrict ourselves to $0\le\Delta\le 1/2$ without losing generality. Ostlund used free-fermion analysis, which is valid for low temperature and Δ close to $1/2$, to show that there are incommensurate phases in this model. This fact makes the model interesting for the study of commensurate-incommensurate phase transitions and hence it has been the focus of considerable theoretical efforts.

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The model (1) has been studied by finite-size scaling methods,[4–8] Monte-Carlo simulation,[9] hierarchical lattice approximation,[10] Monte-Carlo renormalization group[11] and series expansion methods.[12–14] Important analytical predictions using domain-wall arguments and general topological ideas also have been presented.[15–17] In spite of all these efforts, several features remain controversial, for example, the existence of a Lifshitz point at $\Delta \neq 0$ in the phase diagram of this model. Haldane et al.,[15] Schulz,[16] and Von Gehlen and Rittenberg[7] argue against the idea of a Lifshitz point at $\Delta \neq 0$, while Howes,[12] Huse and Fisher,[17] Selke and Yeomans,[9] Duxbury et al.,[4] and Martins and Tsallis[10] are presenting arguments for it. Apart from this controversy over a qualitative feature, there are also uncertainties concerning the nature of various phase transitions in this model.

In order to shed more light on these problems, we used improved effective field theories with clusters to be taken as strips which are infinite in the chiral direction and finite in the non-chiral direction. This treatment is equivalent to separating the original two-dimensional square lattice into many identical decoupled strips with effective fields on their boundaries and treating interactions within them exactly. Effective-field transfer matrix methods[18] can be successfully used in such strip-related calculations.

Obviously, within an improved effective field theory, we have to pay serious attention to

i) how to put the effective fields on the boundary (so as to partially include the effects of the out-of-cluster part of original system) and

ii) how to relate the typical order parameters of the finite-strip system to ones of the original system.

These two aspects determine whether the approximate critical points obtained will be converging to the true ones and how fast the convergence will be. Currently, the most-commonly used effective field theories employ the Gibbs-Bogoliubov free energy variational principle, resulting in the Weiss and Bethe approximations. It has been found that even an infinite chain with effective fields (which are determined from free energy considerations) on the boundaries can qualitatively improve the simple effective field results.[19]

More interestingly, as advocated by Suzuki, it is possible to apply the coherent anomaly method (CAM)[18] to well-chosen sequences of effective field theories. By systematically treating wider and wider strips—i.e. more and more interactions are treated exactly—one obtains better and better approximations to the exact phase diagram of the original physical system and an excellent extrapolation to the exact results can be expected from these successive approximations, if the strips become wide enough.

This paper is organized as follows. In Section 2, we present our analysis of effective field theories based on the Gibbs-Bogoliubov free energy variational principle.[20] In Section 3, we first show how the approximate wavevector-dependent susceptibility is obtained in two series of effective field theories with either Weiss or

Bethe approximations, resulting in two series of Lifshitz point approximants. From these a new series is constructed showing that possibly no Lifshitz point exists at finite (non-zero) chirality.[21] A brief summary is given in Section 4.

2. Effective Field Theory from Free Energy Considerations

The approximate free energy F_{MF} is obtained by the use of the Gibbs-Bogoliubov inequality

$$F \leq F_{\mathrm{MF}} = \min(F_0 + \langle H - H_0 \rangle), \tag{2}$$

where F is the exact free energy of the original system with H being the original Hamiltonian. H_0 is a trial Hamiltonian and F_0 is the exact free energy of the system defined by H_0. The average $\langle \cdots \rangle$ is carried out in the ensemble defined by H_0 and this convention will be used throughout this paper. For boundary spins, it is more convenient to introduce the vector notation

$$\mathbf{S}_{i,j} = \left(\cos \frac{2\pi}{3} n_{i,j},\ \sin \frac{2\pi}{3} n_{i,j} \right). \tag{3}$$

H is given in Eq. (1) and H_0 is defined as follows:

$$\begin{aligned} -\beta H_0 = & \sum_{i,j} K_n \cos \frac{2\pi}{3}(n_{i,j} - n_{i,j+1} + \Delta) \\ & + \sum_{p=0}^{N_s-1} \sum_{k=pN}^{(p+1)N-2} \sum_{j} K_t \cos \frac{2\pi}{3}(n_{k,j} - n_{k+1,j}) \\ & + \sum_{p,p'=0}^{N_s-1} \sum_{k=0}^{L-1} K_t \boldsymbol{\eta}_k \cdot (\mathbf{S}_{pN-1,p'L+k} + \mathbf{S}_{pN,p'L+k}), \end{aligned} \tag{4}$$

where $0 \leq i \leq N_s N - 1$, $0 \leq j \leq N_s L - 1$, periodic boundary conditions are imposed on both directions, and $\beta = 1/k_{\mathrm{B}}T$. The trial Hamiltonian H_0 consists of N_s independent strips of width N and length $N_s L$ with effective boundary fields $\{\boldsymbol{\eta}_j = (\eta_{j1}, \eta_{j2})\}$ having period L to replace the exact interactions between strips. To find a good approximation for the free energy, we use Eq. (2) to find the minimum conditions which $\{\boldsymbol{\eta}_j\}$ should satisfy. The necessary minimum conditions can be simplified as

$$\boldsymbol{\eta}_j = \langle \mathbf{S}_{0,j} \rangle \equiv \mathbf{m}_j. \tag{5}$$

The corresponding approximate free energy per site f_{MF} can be given as

$$f_{\mathrm{MF}} = f_0 + \frac{K_t}{NL\beta} \sum_{j=0}^{L-1} (2\boldsymbol{\eta}_j \cdot \mathbf{m}_j - \mathbf{m}_j \cdot \mathbf{m}_j), \tag{6}$$

where f_0 is the free energy per site of the system defined by H_0 and can be calculated by the effective transfer matrix method.[18,20]

It is easy to see that the effective fields in our trial Hamiltonian are essentially the thermal averages of the strip boundary spins whose configuration can be used to characterize the related phase. Eqs. (5) can be solved by iteration methods and interested readers can consult our full paper[20] for technical details. From physical considerations, we may expect three types of solutions to Eqs. (5), i.e.

i) the disordered solution with ($\boldsymbol{\eta} = \mathbf{0}$),
ii) the ordered solution with ($\boldsymbol{\eta} \neq \mathbf{0}$) which can be obtained by setting $L = 1$ in Eqs. (5), and
iii) modulated solutions with unequal effective fields, i.e. $L > 1$.

The thermodynamically stable phase is the one that gives the absolute minimum free energy for all different solutions with all possible L. Hence, for $0 \leq \Delta \leq 1/2$, we can expect that the disordered solution gives the lowest approximate free energy for the disordered phase and the ordered solution for commensurate phase. In the modulated phase, one of the modulated solutions should give the lowest approximate free energy and the choice of solution may vary from point to point.

The numerical results are summarized as follows. We obtain $\Delta_L(1) \approx 0.3143$, $\Delta_L(2) \approx 0.2883$, $\Delta_L(3) \approx 0.2770$, $\Delta_L(4) \approx 0.2709$ for $K_n = K_t$ and $\Delta_L(1) \approx 0.2258$, $\Delta_L(2) \leq 0.2156$ for $K_n = 10K_t$, where the notation $\Delta_L(N)$ is used to denote the approximate Lifshitz point from the effective field theory for a strip of width N. Hence, we can safely claim that the Lifshitz point $\Delta_L(N)$ located by this approximation is systematically decreased when the width N becomes larger. The result for different ratios of K_n/K_t also coincides with our intuition that the larger K_n/K_t leads to faster convergence. Its possible explanation is discussed in our papers 20 and 21.

As reviewed by Wu,[22] simple mean-field theory predicts a first-order phase transition in the three-state Potts model, which is equivalent to the $\Delta = 0$ three-state chiral clock model. Our effective field theory with finite-width strip also predicts a first-order phase transition for $0 \leq \Delta < \frac{1}{2}$ which is characterized by a sudden change of spin profiles (described by the thermal average of the central-row spins in our effective field theory) due to the discontinuity of the effective fields when the system crosses the critical point. However, this artificial feature of the effective field theory can be overcome by systematically improving the effective field approximation.[20] When $N \to \infty$, these effective fields $\{\eta_j\}$ (which are non-vanishing but have no direct physical meanings in the original problem) should give an infinitesimally small effect on the spin profiles (which should approach zero) and on the specific heat. Hence, the extrapolation of these effective field approximations would be able to give the correct nature of the phase transition, i.e. a continuous phase transition.

3. Effective Field Theory from Susceptibility Considerations

When the system changes from the disordered phase into the incommensurate phase as the temperature is lowered, the peak of the wavevector-dependent susceptibility

also changes into a divergence. Hence, we only have to approximate the wavevector-dependent susceptibility in the disordered phase. We take the trial Hamiltonian of one strip in the disordered regime as follows:

$$-\beta H' = -\beta H + K_t h \sum_j \sum_{i=1}^{N} \cos\left(\frac{2\pi}{3} n_{i,j} - jq\right) + K_t \eta \sum_j \left[\cos\left(\frac{2\pi}{3} n_{1,j} - jq\right) + \cos\left(\frac{2\pi}{3} n_{N,j} - jq\right)\right] \tag{7}$$

where H is a restriction of the exact Hamiltonian taking precisely all its terms within the strip, and where η denotes the amplitude of the modulated effective boundary fields, h the amplitude of the auxiliary external bulk fields, and q the wavevector of the external field and modulated effective boundary fields along the chiral direction. (In the disordered phase and with a weak field condition, we can expect the response of the spin average to be characterized by the same wavevector q because of the symmetry of H and $H'|_{\eta=0,h=0}$ under translation. When we introduce our Weiss and Bethe effective-field approximations, the effective fields should be characterized by this wavevector q as well.) Meanwhile, because most of the previous understanding has come from the study of the Hamiltonian limit, which corresponds to either $K_n/K_t \to 0$ or $K_n/K_t \to \infty$,[6–8,12,14] it is kind of natural for us to keep K_n/K_t general.

Since in all calculations below we take ensemble averages based on H' and often with both η and h being zero, we use $\langle\cdots\rangle$ to denote the statistical average with ensemble based on H' and $\langle\cdots\rangle_0$ to denote $\langle\cdots\rangle|_{\eta=0,h=0}$. For convenience, we also define quantities Q_c and $Q_{\partial,j}$ as follows:

$$Q_c = \begin{cases} \exp\left(\mathrm{i}\frac{2\pi}{3} n_{m+1,0}\right), & \text{if } N = 2m+1, \\ \frac{1}{2}\left[\exp\left(\mathrm{i}\frac{2\pi}{3} n_{m,0}\right) + \exp\left(\mathrm{i}\frac{2\pi}{3} n_{m+1,0}\right)\right], & \text{if } N = 2m, \end{cases} \tag{8}$$

$$Q_{\partial,j} = \frac{1}{2}\left[\exp\left(\mathrm{i}\frac{2\pi}{3} n_{1,j}\right) + \exp\left(\mathrm{i}\frac{2\pi}{3} n_{N,j}\right)\right]. \tag{9}$$

These quantities have a direct interpretation: Q_c is the spin in the middle row of the 0-th column, if the number of rows N is odd. If the number of rows N is even, we take the average over the two middle rows. $Q_{\partial,j}$ is the average of the two spins in the boundary rows $i = 1$ and $i = N$ of the j-th column ($j = -\infty, \cdots, \infty$).

We put the self-consistency conditions

$$\langle Q_c \rangle = \eta \qquad \text{for Weiss approximation,} \tag{10}$$

$$\langle Q_c \rangle = \langle Q_{\partial,0} \rangle \quad \text{for Bethe approximation.} \tag{11}$$

The wavevector-dependent susceptibility has a peak located at q_m which gives an approximation to the characteristic wavevector of the corresponding correlation function. By some tedious calculation, the critical point that demarcates the

paramagnetic-incommensurate phase transition can be located from

$$\min_q \left(1 - K_c \sum_j \langle Q_c Q^*_{\partial,j}\rangle_0 \exp(ijq)\right) = 0 \quad \text{for Weiss approximation,} \tag{12}$$

$$\min_q \sum_j \left(\langle Q_{\partial,0} Q^*_{\partial,j}\rangle_0 - \langle Q_c Q^*_{\partial,j}\rangle_0\right) \exp(ijq) = 0 \quad \text{for Bethe approximation,} \tag{13}$$

where the minimum condition is over all q. The corresponding q_m will give an approximation to the wavevector q_c, characteristic of the correlation function at the phase transition point. Here and in the following we write $K \equiv K_t$ and $K_c \equiv K_{tc}$, its value at the critical point separating the disordered and modulated phases. K_n and K_t vary proportionally.

In both approximations, the susceptibility near K_c ($K < K_c$) can be presented in the form

$$\chi = \overline{\chi}/(\frac{K_c}{K} - 1). \tag{14}$$

where $\overline{\chi}$ is the coherent anomaly coefficient and has been worked out for both cases.[21]

From the above, we can obtain two series of approximations. However, both series are short and hence difficult to extrapolate. To circumvent this problem, we construct a new extrapolation method as follows.

If there exist two sequences $\{a(n)\}$ and $\{b(n)\}$, which satisfy

i) $\lim_{n\to\infty} a(n) = c$, $\lim_{n\to\infty} b(n) = c$ and $a(n)$, $b(n) \neq c$ for any n,
ii) $\lim_{n\to\infty} \big(a(n+\delta n) - a(n)\big)/\big(b(n+\delta n) - b(n)\big)$ exists and is not 1,

it is possible to construct a third sequence $\{c(n)\}$ with $\lim_{n\to\infty} c(n) = c$ by

$$c(n) = \frac{a(n+\delta n)b(n) - a(n)b(n+\delta n)}{a(n+\delta n) - a(n) - b(n+\delta n) + b(n)}. \tag{15}$$

Under certain conditions, we can expect that the sequence $\{c(n)\}$ will converge faster than either $\{a(n)\}$ or $\{b(n)\}$.

We find that this new construction works very well for the square lattice Ising model and Potts cases with various ratios of K_n/K_t.[21] Here we only present the Potts model results, i.e. the case with $\Delta = 0$ and $K_n = K_t$, in Table 1, where

Table 1. Table of T_b, T_w and T_n.

N	3	4	5	6	7
T_b	1.56208	1.55004	1.54073	1.53471	1.52965
T_w	1.65702	1.62624	1.60251	1.58794	1.57563
T_n	1.5010	1.4992	1.4974		

critical temperature $T_b(N)$ is obtained by Bethe approximation, $T_w(N)$ by Weiss approximation, N being the width of the finite strip, and $T_n(N)$ is obtained by Eq. (15) with $\delta N = 2$. The exact value for $N = \infty$ is $T_c^* = 1.4925$. We clearly

see good convergence of series $\{T_n(N)\}$. We have also used this construction to find the range of critical temperatures for $\Delta \neq 0$.[21] More interestingly, we use this construction to find the characterizing wavevector along the critical line.

As is well-known,[4] we should be able to get phase transition information through the analysis of the wavevector at the phase transition point. If a Lifshitz point Δ_L exists at finite chirality, the characterizing wavevector along the critical line should vanish for $\Delta \leq \Delta_L$. Although we are not sure how this Lifshitz point Δ_L will depend on K_n/K_t, old works[4,11,12] indicate that there is no big dependence of Δ_L on K_n/K_t.

Two cases with $K_n = 10K_t$ and $K_n = 100K_t$ at $\Delta = 0.05$ have been studied. The results for the two cases are similar, so we only present in Table 2 the results for the case with $\Delta = 0.05$ and $K_n = 100K_t$, where the reduced critical wavevector

Table 2. Table of $\hat{q}_b$, $\hat{q}_w$ and $\hat{q}_n$.

N	3	4	5	6	7
$\hat{q}_w$	0.0463680	0.0402710	0.0353594	0.0320544	0.0291924
$\hat{q}_b$	0.0362354	0.0314054	0.0276127	0.0250293	0.0228355
$\hat{q}_n$	−0.00038	0.00069	0.00099		

is defined by $\hat{q} = 3q/(2\pi\Delta)$. Here, $\hat{q}_b(N)$ is obtained by Bethe approximation and $\hat{q}_w(N)$ by Weiss approximation, where N is the width of the finite strip, and $\hat{q}_c(N)$ is obtained by Eq. (15) with $\delta N = 2$. These calculations for the wavevector need an accuracy of 10^{-8} for q. Higher accuracy will be needed for smaller Δ and our numerical values would not have been reliable enough then.

Although we only have three members in this new sequence $\{\hat{q}_n(N)\}$ and we cannot make a very conclusive case, it looks very tempting to say that this sequence will converge to the true $\hat{q}_c$ from below. Compared with previous results for Δ_L to be around 0.25 to 0.40,[4,11,12] we have $\Delta_L < 0.05$. Hence, we may conclude that even for a very small Δ the wavevector at the transition point is non-zero. This means that the transition should be from the paramagnetic to the incommensurate phase and that possibly no Lifshitz point exists at finite chirality at all.

4. Summary

In the above sections, we have used two different methods to approach the problem whether a Lifshitz point exists in the two-dimensional classical three-state chiral clock model at finite non-zero chirality. The first method gives more information about the general phase diagram, whereas the second method seems to more reliably determine the boundary of the disordered phase. However, both extrapolations together consistently indicate that most likely no Lifshitz point exists in this model at finite non-zero chirality. A study of somewhat wider strips on more powerful computers may take away all remaining doubt.

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STOCHASTIC DESCRIPTION OF AGGLOMERATION AND GROWTH PROCESSES IN GLASSES

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We show how growth by agglomeration can be described by means of algebraic or differential equations which determine the evolution of probabilities of various local configurations. The *minimal fluctuation* condition is used to define vitrification. Our methods have been successfully used for the description of glass formation.

1. Introduction

In a series of papers published during the past ten years,[1–4] new models of growth by agglomeration of smaller units have been elaborated, and applied to many important physical systems, such as quasicrystals,[5] fullerenes,[6,7] and oxide and chalcogenide glasses.[8–11] Here we shall present the main ideas on which these models are based, and briefly discuss the latest developments.

In order to make our presentation concise, the example we choose is the simplest covalent network glass known to physicists, the binary chalcogenide glass $As_xSe_{(1-x)}$, where x is the concentration of arsenic atoms in the basic glass-former, which in this case is pure selenium. The generalization to other covalent networks, e.g. $Ge_xSe_{(1-x)}$, is quite straightforward. These glasses (in the form of thin and elastic foils) are used in photocopying devices.

Whether the formation of a solid network of atoms or molecules occurs in a more or less rapidly cooled liquid, or as vapor condensation on a cold support, the most important common feature of these processes is progressive agglomeration of small and mobile units (which may be just single atoms, or stable molecules, or even small clusters already present in the liquid state) into an infinite stable network, whose topology can no longer be modified unless the temperature is raised again, leading to the inverse (melting or evaporation) process.

To describe such an agglomeration with all geometrical and physical parameters, such as bond angles and lengths, and the corresponding chemical and mechanical

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energies stored in each newly formed bond, is beyond the possibilities of any reasonable model. This is why stochastic theory is an ideal tool for the description of random agglomeration and growth processes. Instead of reconstructing all local configurations, it takes into account only the *probabilities* of them being found in the network, and then the probabilities of higher order, corresponding to local correlations. This is achieved by using the *stochastic matrix* technique. A *stochastic matrix* $\mathbf{M}$ represents an operator transforming given finite distribution of probabilities , $[p_1, p_2, ..., p_N]$, into another distribution of probabilities, $[p_1', p_2', ..., p_N']$. It follows immediately that such a matrix must have only real non-negative entries, each column summing up to 1.

The algebraic properties of such matrices are very well known. The main feature that we shall use here is the fact that any stochastic matrix has at least one eigenvalue equal to 1. The remaining eigenvalues have their absolute values always less than 1. This means that if we continue to apply a stochastic matrix to any initial probability distribution, after some time only the distribition corresponding to the unit eigenvalue will remain, all other contributions shrinking exponentially. This enables us to find the asymptotic probability distribution.

In what follows, we identify these probability distributions with stable or metastable states of the system, fixing the statistics of characteristic sites in the network. Taking into account Boltzmann factors (with chemical potentials responsible for the formation of bonds), we are able to find the glass transition temperature in various compounds. In particular, one is able to predict the initial slope of the curve $T_g(c)$, i.e. the value of $(dT_g/dc)_{c=0}$.[12,13]

2. Stochastic matrix describing cluster agglomeration

Consider a binary selenium-arsenic glass, in which selenium is the basis glass former, and arsenic is added as modifier (although its concentration can be as high as 30%). The chemical formula denoting this compound is $As_cSe_{(1-c)}$, where c is the As concentration. In a hot liquid, prior to solidification, the basic building blocks that agglomerate are just selenium and arsenic atoms, indicated respectively by (—o—) and  . When the temperature goes down, clusters of atoms start to appear everywhere, growing by agglomeration of new atoms on their rim. Consider a growing cluster: one can distinguish three types of situations (we shall call them *"sites"*) on the cluster's rim. The concentration of free As atoms in the liquid will be called c and that of Se, $(1-c)$.

Two choices are possible for constructing the states and transition matrix (see Ref. 14). There are three possible kinds of sites: a selenium atom with one unsaturated bond, and an As atom presenting one or two free bonds; these are indicated by $x =$ o—, $y =$  and $z =$ •—. To each site one of the two basic cells can attach itself, reproducing one of the initial configurations, in the specific combinations shown in the next column of the Figure 1. The attachment of *one single basic cell*, or the saturation of one single bond, is a step in the evolution. In the second choice,

each step is obtained by the complete saturation of all the bonds at the rim, so that only two types of sites (denoted by x and y) are seen on cluster's rim, assuming that the growth is of dendritic type (no small rings present). It can be shown[14] that the two approaches lead to the same results, which may be considered as a proof of the ergodicity of the proposed model. We shall choose the second version of the model for the sake of simplicity. In this case, we can take into account only the x and y-type sites, because the z-type sites transform after the next agglomeration step into an x or y type site. The elementary step in the agglomeration process, described by he transition matrix, corresponds now to the complete saturation of all the available free bonds on the rim. This is represented in Figure 1 :

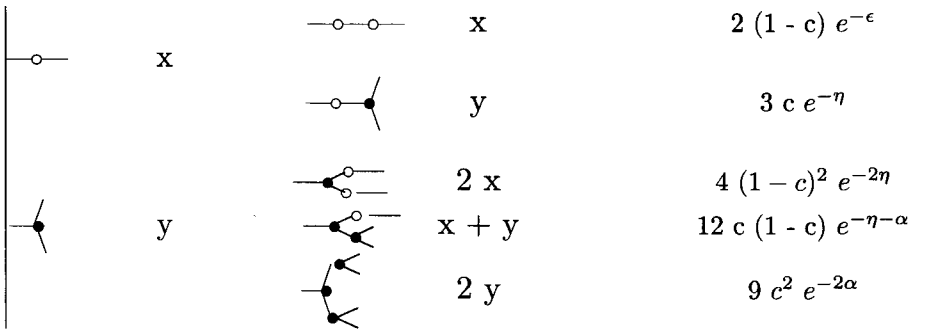

Figure 1: *States, steps and un-normalized probability factors .*

Observing that from the site z only the sites of x and y type can be produced, we can forget it and consider the dendritic growth with only two types of sites appearing all the time. Given an arbitrary initial state (p_x, p_y), the new state results from taking into account all possible ways of saturating the bonds of the previous state's sites by the available external atoms. The un-normalized probability factors are displayed in the Figure. The non-normalized probability factors can be arranged in a matrix

$$\begin{pmatrix} 2(1-c)e^{-\epsilon} & 4(1-c)^2e^{-2\eta} \\ 8(1-c)^2e^{-2\eta} + 12c(1-c)e^{-\eta-\alpha} & 12c(1-c)e^{-\eta-\alpha} + 18c^2e^{-2\alpha} \end{pmatrix} \tag{1}$$

The normalized transition matrix is written as

$$M = \begin{pmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{pmatrix} = \begin{pmatrix} M_{xx} & 1 - M_{yy} \\ 1 - M_{xx} & M_{yy} \end{pmatrix} \tag{2}$$

where the entries are obtained by normalizing the columns of the matrix (1).

$$M_{xx} = \frac{2(1-c)\xi}{2(1-c)\xi + 3c}, \quad \text{and} \quad M_{yy} = \frac{3c\mu}{2(1-c) + 3c\mu} \tag{3}$$

where we have introduced the abbreviated notation $\xi = e^{\eta-\epsilon}$ and $\mu = e^{\eta-\alpha}$.

The eigenvalues of this matrix are 1 and $M_{xx} - M_{yy} = M_{xy} - Myx$. and the stationary eigenvector is

$$\begin{pmatrix} p_x^\infty \\ p_y^\infty \end{pmatrix} = \frac{1}{M_{xy} + M_{yx}} \begin{pmatrix} M_{xy} \\ M_{yx} \end{pmatrix}, \tag{4}$$

It can be seen from Figure 1 that on the surface of an average cluster, p_x is the Se concentration and p_y is the As concentration. Now, the high homogeneity exhibited by known glass structures suggests that even in relatively small clusters, deviations from the average modifier concentration c must be negligible. Thus, *in the bulk*, the As concentration should be equal to c. Therefore, the condition of minimal fluctuations in the bulk concentration can be interpreted as the glass transition condition. This means that the asymptotic state is fixed by the external concentration, therefore the above eigenvector must be equal to the average distribution vector $(1-c, c)$. The solutions are $c = 0$, $c = 1$ and the nontrivial one

$$c = \frac{M_{yx}}{M_{xy} + M_{yx}} = \frac{6 - 4\xi}{12 - 4\xi - 9\mu} \,. \tag{5}$$

This equation can be checked against experiment. For example, we can evaluate the derivative $\frac{\partial T}{\partial c} = \left(\frac{\partial c}{\partial T}\right)^{-1}$ for a given value of c. In particular, as $c \to 0$, we can neglect the As–As bond creation (equivalent to putting $\mu = 0$ in (5)), to get

$$\left[\frac{\partial T}{\partial c}\right]_{c=0} = \frac{T_{g_0}}{\ln(3/2)} \,,$$

(where T_{g_0} is the glass transition temperature of pure Se). This is the present–case expression of the general formula given by the stochastic approach, the fraction $(3/2)$ being replaced by (m'/m), where m and m' are the valences of the basic glass former and of the modifier), remaining in very good agreement with the experimental data (see Refs. 15–17).

3. Low concentration limit.

The above scheme can be easily generalized to the case of arbitrary valence, say m_A and m_B. In that case, the stochastic 2×2 matrix has the same form as (2), but with the entries given by

$$M_{xx} = 1 - M_{yx} = \frac{m_A(1-c)\xi}{m_A(1-c)\xi + m_B c} \,, \qquad M_{xy} = 1 - M_{yy} = \frac{m_A(1-c)}{m_A(1-c) + m_B c\mu} \,,$$

The asymptotic probability has the same form as before, as well as the zero fluctuation condition relating c with T (interpreted as the glass transition temperature). The derivative of c with respect to the temperature T gives the "magic formula"

$$\frac{dc}{dT} = \frac{1}{T} \frac{(\frac{m_A}{m_B} - \mu)\,\xi ln\xi - (\frac{m_B}{m_A} - \xi)\,\mu ln\mu}{[(1 - \frac{m_A}{m_B}\xi) + (1 - \frac{m_B}{m_A}\mu)]^2} \tag{6}$$

where we used the fact that $\frac{d\xi}{dT} = -\frac{1}{T}\,\xi\, ln\xi$, and $\frac{d\mu}{dT} = -\frac{1}{T}\,\mu\, ln\mu$. This defines the slope of the function $T_g(c)$, which is an important measurable quantity :

$$\frac{dT_g}{dc} = T_g \frac{[(1 - \frac{m_A}{m_B}\,\xi) + (1 - \frac{m_B}{m_A}\,\mu)]^2}{(\frac{m_A}{m_B} - \mu)\,\xi ln\xi - (\frac{m_B}{m_A} - \xi)\,\mu ln\mu} \tag{7}$$

The initial slope, at $c = 0$, is of particular interest. Its expression is very simple, taking into account that when $c = 0$, we have also $\xi = \frac{m_B}{m_A}$, which leads to

$$\left[\frac{dT_g}{dc}\right]_{c=0} = \frac{T_{g0}\,(1 - \frac{m_B}{m_A}\,\mu)}{ln(\frac{m_B}{m_A})} \tag{8}$$

Its value has been checked against the experiment very successfully, in more than 30 different compounds. In some cases the formula does not seem to work well; usually it comes from the change of valence of certain atoms provoked by the influence of the surrounding substrate.

One could be worried about the apparent singularity in this formula when $m_A = m_B$, i.e. when one deals with a mixture of two different glass formers with the same coordination number. It is not difficult to show that also in such a case a reasonable limit can be defined, as has been recently suggested by M. Micoulaut.[19] As a matter of fact, suppose that the glass transition temperature of the pure glass-former A is T_{g0}, and that of the pure glass-former B is T_{g1}. We can re-write our minimal fluctuation condition in a very symmetric manner, invariant with respect to the simultaneous substitution $m_A \leftrightarrow m_B$, $c \leftrightarrow (1-c)$ and $\xi \leftrightarrow \mu$:

$$c(1-c)\,[(1-c)\,(1 - \frac{m_A}{m_B}\,\xi) - c\,(1 - \frac{m_B}{m_A}\,\mu)] = 0 \tag{9}$$

Obviously, the "pure states" $c = 0$ or $c = 1$ represent stationary solutions of (9) and can be factorized out. The non-trivial condition for the glass forming is thus

$$(1-c)\,[1 - \frac{m_A}{m_B}\,\xi] - c\,[1 - \frac{m_B}{m_A}\,\mu] = 0 \tag{10}$$

Now, using the limit conditions at $c \to 0$, $T_g = T_{g0}$ and $c \to 1$, $T_g = T_{g1}$, and introducing the generalized Boltzmann factors with the energy barriers for corresponding bond creations as E_{AA}, E_{AB} and E_{BB}, we can write

$$E_{AB} - E_{AA} = k\,T_{g0}\,ln(\frac{m_B}{m_A})\,, \quad E_{AB} - E_{BB} = k\,T_{g1}\,ln(\frac{m_A}{m_B}), \tag{11}$$

so that the expressions ξ and μ at the arbitrary temperature T can be written as

$$\xi(T) = e^{\frac{E_{AB}-E_{AA}}{T_{g0}}\cdot\frac{T_{g0}}{T}} = (\frac{m_B}{m_A})^{\frac{T_{g0}}{T}}\,; \quad \mu(T) = e^{\frac{E_{AB}-E_{BB}}{T_{g1}}\cdot\frac{T_{g1}}{T}} = (\frac{m_A}{m_B})^{\frac{T_{g1}}{T}}. \tag{12}$$

Substituting these expressions into (7) and taking the limit $c \to 0$, we get

$$\frac{dT_g}{dc}\,|_{c=0} = \frac{T_{g0}[1 - (\frac{m_B}{m_A})^{\frac{T_{g0}-T_{g1}}{T_{g0}}}]}{ln(\frac{m_B}{m_A})} \tag{13}$$

It is easy to see now that even when $m_A = m_B$, this formula has a well defined limit. Indeed, if we first set $\frac{m_B}{m_A} = 1 + \epsilon$, and then develop the numerator and the denominator of the above equation in powers of ϵ, then in the limit when $\epsilon \to 0$ we arrive at a simple linear dependence which is in agreement with common sense and with experiment as well, namely

$$\frac{dT_g}{dc}\,|_{c=0} = T_{g1} - T_{g0} \tag{14}$$

This formula is also confirmed by many experiments, e.g. performed on selenium-sulfur mixtures (where $m_A = m_B = 2$). The deviations from the linear law (14) observed in the $Se - Te$ binary glass are explained by the fact of the chemical properties of tellurium, which changes its valence from 2 to 3 in presence of selenium.

4. The effect of rapid cooling

An interesting extension of this model is obtained when we take into account the effects of rapid cooling, i.e. when the time derivative of the temperature can no longer be neglected. The treatment of this problem was suggested in Ref. 20, and has been solved quite recently.[21]

Consider the agglomeration process defined by the above stochastic matrix, $\vec{p}' = M\vec{p}$, with $\vec{p}$ representing a normalized column (a "vector") with two entries, p_x and $p_y = 1 - p_x$. After one agglomeration step, representing on the average one new layer formed on the rim of a cluster, we can write

$$\Delta\vec{p} = \vec{p}' - \vec{p} = (M - \mathbf{1})\,\vec{p} \tag{15}$$

Let us introduce a symbolic variable s defining the progress of the agglomeration process; obviously, $s(t)$ should be a monotonically increasing function during the glass transition. If the temperature variation is so slow that the derivative $dT/dt = (dT/ds)(ds/dt)$ can be neglected (which is often called the *annealing* of glass), the master equation of our model can be written as

$$\Delta\vec{p} = \frac{\partial\vec{p}}{\partial s}\,\Delta s = (M - \mathbf{1})\,\vec{p}\,\Delta s$$

where the variation Δs represents one complete agglomeration step. If we want to use real time t as an independent parameter, we should write

$$\frac{d\vec{p}}{dt} = \frac{\Delta\vec{p}}{\Delta s}\frac{ds}{dt} = \tau^{-1}\,\frac{\Delta\vec{p}}{\Delta s} = \frac{1}{\tau}\,(M - \mathbf{1})\,\vec{p} \tag{16}$$

We have introduced here the new entity $\tau = (ds/dt)^{-1}$ which can be interpreted as the average time needed to complete a new layer in any cluster, or alternatively, the time needed for an average bond creation. Now, if the temperature varies rapidly enough, the matrix M can no longer be considered as constant. The equation (16) must be modified according to the well known "moving target" principle. That is, the *total* derivative of $\vec{p}$ with respect to t should read:

$$\frac{d\vec{p}}{dt} = (M - \mathbf{1})\,\frac{ds}{dt}\,\vec{p} + \frac{\partial M}{\partial T}\,\frac{dT}{dt}\,\vec{p} = \frac{d\vec{p}}{dt} = \left[\frac{1}{\tau}\,(M - \mathbf{1}) + q\,\frac{\partial M}{\partial T}\right]\vec{p} \tag{17}$$

where we supposed linear dependence of the temperature on time, so that the derivative dT/dt can be denoted by constant cooling rate q. In the two-dimensional case only one component of $\vec{p}$ is independent, because $p_x + p_y = 1$. Let us choose p_y (whose asymptotic value should be equal to c) as independent variable. Then (17) will reduce to the single equation :

$$\frac{dp_y}{dt} = \frac{1}{\tau}\left[(M_{yy} - 1)\,p_y + M_{yx}\,(1 - p_y)\right] + q\left[\frac{\partial M_{yy}}{\partial T}\,p_y + \frac{\partial M_{yx}}{\partial T}\,(1 - p_y)\right] \tag{18}$$

where we have used the fact that $p_x = 1 - p_y$, $M_{xx} = 1 - M_{yx}$ and $M_{yy} = 1 - M_{xy}$.

What remains is just simple algebra. After a few operations we find the asymptotic value of p_y, denoted p_y^∞, obtained when we set $dp_y/dt = 0$:

$$p_y^\infty = \frac{M_{yx} + \tau q\,(\frac{\partial M_{yx}}{\partial T})}{(M_{xy} + M_{yx}) + \tau q(\frac{\partial (M_{xy} + M_{yx})}{\partial T})} \tag{19}$$

As in the former case, we define the glass transition temperature by solving the zero-fluctuation condition $p_y^\infty = c$. The quasi-equilibrium condition thus obtained can be written in a form displaying an apparent symmetry between the two ingredients ("A" and "B") of binary glass. As in the previous case (when $q = 0$), the limit values $c = 0$ and $c = 1$ represent stationary solutions, which is obvious (no local fluctuations of concentration c are possible when there is no ingredient other than A or B atoms alone). After factorizing out $c(1-c)$, we get

$$\frac{m_B}{m_A(1-c)\xi + m_B c} - \frac{m_A}{m_A(1-c) + m_B c\mu} =$$

$$\frac{\tau q}{T}\, m_A m_B \left[\frac{c\,\mu\, ln\mu}{[m_A(1-c) + m_B c\mu]^2} - \frac{(1-c)\,\xi\, ln\xi}{[m_A(1-c)\,\xi + m_B c]^2} \right] \tag{20}$$

where we have used the fact that $\frac{\partial (ln\,\xi)}{\partial T} = -\frac{ln\,\xi}{T}$, $\frac{\partial (ln\,\mu)}{\partial T} = -\frac{ln\,\mu}{T}$. The above formula seems quite cumbersome, but it become much simpler in the low concentration limit, $c \to 0$ Close to $c = 0$ we get

$$\frac{m_B}{m_A} - \xi + \frac{\tau q}{T}\frac{m_B}{m_A}\, ln\,\xi = 0 \tag{21}$$

(quite obviously, in the limit $c \to 1$ one gets the same formula switching m_A with m_B and replacing ξ by μ). Replacing ξ by the expression (12), we arrive at :

$$\left[1 - \left(\frac{m_B}{m_A} \right)^{\frac{T_{g0} - T}{T}} \right] + (\frac{\tau q}{T})\frac{T_{g0}}{T}\, ln\,(\frac{m_B}{m_A}) = 0\,. \tag{22}$$

It is easy to see that independently of the ratio m_B/m_A, for temperatures T *above* T_{g0} we must have $q < 0$, and vice-versa, during rapid cooling the glass transition occurs at the temperature $T > T_{g0}$.

The dimensionless combination $(\tau\, q)/T$ defines the quenching rate as the product of $(1/T)(dT/dt) = d(lnT)/dt$ by the time constant τ, characterizing the kinetics of the agglomeration process, i.e. the average time it takes to create a new bond. It may depend weakly on the temperature, but for the sake of simplicity suppose it is constant. It can be determined by comparing formula (22) with the experimental data. To take an example, let us again consider the selenium-arsenic glass at $c \to 0$ (almost pure selenium with a small addition of As). We know that in this case $T_g \to T_{g0} = 318^0 K$. The formula (22) then gives the quasi-linear dependence of $\Delta T = T - T_{g0}$ on the quenching rate q: for $T_g = 328^0 K$ (i.e. $\Delta T = 10^0 K$) we get $\tau\, q = -10.38$; for $T_g = 338^0 K$ (i.e. $\Delta T = 20^0 K$) we get $\tau\, q = -21.51$; for $T_g = 348^0 K$ (i.e. $\Delta T = 30^0 K$) we get $\tau\, q = -32.26$, and so forth.

Finally, if we want to establish the formula for a pure glass-former, without any modifier, we should take the limit $(m_A/m_B) \to 1$ and $\mu \to \xi$; we then get

$$\frac{T - T_{g0}}{T} + (\frac{\tau q}{T}) \frac{T_{g0}}{T} = 0 \quad \text{or} \quad T - T_0 = \Delta T_g = -(\tau q) \frac{T_{g0}}{T}. \tag{23}$$

Eventually, the deviations from this simple dependence may indicate that the characteristic time τ depends on T. This can shed more light on the agglomeration kinetics in various glass-forming liquids. More details can be found in Refs. 18, 21.

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FUSION CONSTRUCTION OF THE VERTEX OPERATORS IN HIGHER LEVEL REPRESENTATION OF THE ELLIPTIC QUANTUM GROUP*

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After a short summary on the elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ and the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, we present a free field representation of the Drinfeld currents and the vertex operators (VO's) in the level k. We especially demonstrate a construction of the higher spin type I VO's by fusing the spin 1/2 type I VO's and fix a rule of attaching the screening current $S(z)$ associated with the q-deformed $\mathbb{Z}_k$-parafermion theory. As a result we get a free field representation of the higher spin type I VO's which commutation relation by the fused Boltzmann weight coefficients is manifest.

1. Elliptic Quantum Group and Elliptic Algebra

1.1. *Elliptic Quantum Group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ [2]*

The face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ is a quasi-Hopf algebra $(\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2), \Delta_\lambda, \epsilon, S, \Phi(\lambda), \alpha, \beta, \mathcal{R}(\lambda))$ obtained as a deformation of the Hopf algebra $(U_q(\widehat{\mathfrak{sl}}_2), \Delta, \epsilon, S, \mathcal{R})$ by the face type twistor $F(\lambda)$ $(\lambda \in \mathfrak{h})$ satisfying the shifted cocycle condition[a]

$$F^{(12)}(\lambda)(\Delta \otimes \mathrm{id})F(\lambda) = F^{(23)}(\lambda + h^{(1)})(\mathrm{id} \otimes \Delta)F(\lambda). \tag{1.1}$$

The "deformation" means that $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2) \cong U_q(\widehat{\mathfrak{sl}}_2)$ as an associative algebra, but the coalgebra structure is deformed in the sense $\Delta_\lambda(x) = F(\lambda)\Delta(x)F^{-1}(\lambda)$ $\forall x \in U_q(\widehat{\mathfrak{sl}}_2)$, $\mathcal{R}(\lambda) = F^{(21)}(\lambda)\mathcal{R}F^{(12)-1}(\lambda)$, etc. The universal R matrix $\mathcal{R}(\lambda)$ satisfies the dynamical Yang–Baxter equation.

$$\mathcal{R}^{(12)}(\lambda + h^{(3)})\mathcal{R}^{(13)}(\lambda)\mathcal{R}^{(23)}(\lambda + h^{(1)}) = \mathcal{R}^{(23)}(\lambda)\mathcal{R}^{(13)}(\lambda + h^{(2)})\mathcal{R}^{(12)}(\lambda). \tag{1.2}$$

¿From this, we obtain the dynamical RLL relation which characterizes $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$

$$\begin{aligned} &R^+_{VW}(z_1/z_2, \lambda + h)L^+_V(z_1, \lambda)L^+_W(z_2, \lambda + h^{(1)}) \\ &\quad = L^+_W(z_2, \lambda)L^+_V(z_1, \lambda + h^{(2)})R^+_{VW}(z_1/z_2, \lambda). \end{aligned} \tag{1.3}$$

*A short report on a part of the work done with Robert Weston.[1]
[a]We follow the notation in Ref. 2.

Here $L_V^+(z,\lambda) = (\pi_{V,z} \otimes \mathrm{id})\,\mathcal{R}^+(\lambda)$, $R_{VW}^+(z_1/z_2,\lambda) = (\pi_{V,z_1} \otimes \pi_{W,z_2})\,\mathcal{R}^+(\lambda)$, $\mathcal{R}^+(\lambda) = q^{c\otimes d + d\otimes c}\mathcal{R}(\lambda)$ in the evaluation representation $(\pi_{V,z}, V_z)$.

Hereafter we use the parameterization $\lambda = (r-c+2)d + (P+1)\frac{1}{2}h_1$ and $\lambda + h = (r+2)d + (P+h_1+1)\frac{1}{2}h_1$ $(h_1 \in \bar{\mathfrak{h}})$. Then the two dimensional representation $(\pi_{V^{(1)},z_1} \otimes \pi_{V^{(1)},z_2})$ of the R matrix, up to gauge transformation, is given by

$$R^+(z,P+h_1) \equiv R^+_{V^{(1)}V^{(1)}}(z,\lambda+h) = \rho^+(u)\begin{pmatrix} 1 & & & \\ & b(u,P+h_1) & c(u,P+h_1) & \\ & \bar{c}(u,P+h_1) & \bar{b}(u,P+h_1) & \\ & & & 1 \end{pmatrix},$$

$$R^{*+}(z,P) \equiv R^+_{V^{(1)}V^{(1)}}(z,\lambda) = R^+(z,P)|_{r\to r^*}.$$

Here $z = q^{2u}$, $p = q^{2r}$, $r^* = r-c$, $p^* = q^{2r^*}$ and $\rho^+(u) = z^{\frac{1}{2r}}\frac{\{pq^2z\}^2}{\{pz\}\{pq^4z\}}\frac{\{z^{-1}\}\{q^4z^{-1}\}}{\{q^2z^{-1}\}^2}$, $\{z\} = (z;p,q^4)_\infty$ and $b(u,s) = \frac{[s+1][s-1][u]}{[s]^2[1+u]}$, $c(u,s) = \frac{[s+u][1]}{[s][1+u]}$, $\bar{c}(u,s) = \frac{[s-u][1]}{[s][1+u]}$, $\bar{b}(u,s) = \frac{[u]}{[1+u]}$. The symbol $[u]$ denotes the Jacobi theta function $[u] = q^{\frac{u^2}{r}-u}\frac{\Theta_p(q^{2u})}{(p;p)_\infty^3}$, $\Theta_p(z) = (z;p)_\infty(p/z;p)_\infty(p;p)_\infty$, $(z;p_1,\cdots,p_k)_\infty = \prod_{n_1,\cdots,n_k\geq 0}(1-zp_1^{n_1}\cdots p_k^{n_k})$. We also use $[u]^* = [u]|_{r\to r^*}$. These R matrices are nothing but the Boltzmann weight of the Andrews–Baxter–Forrester (ABF) model. Hence one can regard $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ as a central extension of Felder's elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$.

1.2. *The elliptic algebra* $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

The algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is an elliptic analogue of the algebra $U_q(\widehat{\mathfrak{sl}}_2)$ in the formulation via the Drinfeld currents. Our currents $E(u), F(u), K(u)$ satisfy the following relations.[3]

$$\begin{aligned}
&K(u)K(v) = \rho(u-v)K(v)K(u),\\
&K(u)E(v) = \frac{[u-v+\frac{1-r^*}{2}]^*}{[u-v-\frac{1+r^*}{2}]^*}E(v)K(u),\\
&K(u)F(v) = \frac{[u-v-\frac{1+r}{2}]}{[u-v+\frac{1-r}{2}]}F(v)K(u),\\
&[u-v-1]^*E(u)E(v) = [u-v+1]^*E(v)E(u),\\
&[u-v+1]F(u)F(v) = [u-v-1]F(v)F(u),\\
&[E(u),F(v)] = \frac{1}{q-q^{-1}}\left(\delta(q^{-c}\frac{z}{w})H^+(v+\frac{c}{4}) - \delta(q^c\frac{z}{w})H^-(v-\frac{c}{4})\right).
\end{aligned}$$

Here $\rho(u) = \rho^{+*}(u)/\rho^+(u)$, $\rho^{+*}(u) = \rho^+(u)|_{r\to r^*}$ and

$$H^\pm(u) = \kappa K\left(u \pm \frac{2r-c}{4} + \frac{1}{2}\right)K\left(u \pm \frac{2r-c}{4} - \frac{1}{2}\right)$$

with $\kappa = \lim_{z\to q^{-2}}\frac{\xi(z;p^*,q)}{\xi(z;p,q)}$, $\xi(z;p,q) = \frac{(q^2z;p,q^4)_\infty(pq^2z;p,q^4)_\infty}{(q^4z;p,q^4)_\infty(pz;p,q^4)_\infty}$.

Let us introduce the half currents $K^+(u)$, $E^+(u)$ and $F^+(u)$ by

$$K^+(u) := K(u + \tfrac{r+1}{2}), \tag{1.4}$$

$$E^+(u) := a^* \oint_{C^*} \frac{dz'}{2\pi i z'} E(u' + \frac{c+1}{2}) \frac{[u-u'+\frac{1}{2}-P]^*[1]^*}{[u-u'-\frac{1}{2}]^*[P-1]^*}, \tag{1.5}$$

$$F^+(u) := a \oint_{C} \frac{dz'}{2\pi i z'} F(u' - \frac{1}{2}) \frac{[u-u'-\frac{1}{2}+P+h_1][1]}{[u-u'+\frac{1}{2}][P+h_1-1]}. \tag{1.6}$$

Here C^* : $|p^*q^{-1}z| < |z'| < |q^{-1}z|$, C : $|pqz| < |z'| < |qz|$. a and a^* are the normalization constants satisfying $\frac{aa^*[1]^*}{q-q^{-1}} = 1$.

Theorem 1.1: Define the L-operator $\widehat{L}^+(z) \in \mathrm{End}\, V_z^{(1)} \otimes U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by

$$\widehat{L}^+(z) := \begin{pmatrix} 1 & F^+(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K^+(u-1) & 0 \\ 0 & K^+(u)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ E^+(u) & 1 \end{pmatrix}.$$

Then we have the following RLL-relation.

$$R^+(z_1/z_2, P+h_1)\, \widehat{L}^+(z_1)\widehat{L}^+(z_2) = \widehat{L}^+(z_2)\widehat{L}^+(z_1) R^{*+}(z_1/z_2, P). \tag{1.7}$$

¿From this, one can recover the dynamical RLL relation (1.3) in $V = W = V^{(1)}$ by introducing the new L-operator $L^+(z,P) := \widehat{L}^+(z) \begin{pmatrix} e^{-Q} & 0 \\ 0 & e^{Q} \end{pmatrix}$. In fact, $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ can be regarded as an extension of the algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ by an extra element e^Q and imposing the commutation relation $[P, e^Q] = -e^Q$.[4] The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ hence provides an alternative formulation of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ via the Drinfeld currents.

1.3. *The vertex operators of* $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

¿From the spin $l/2$ type I and II intertwining operators $\Phi^{(\nu,\mu)}_{V^{(l)}}(z)$, $\Psi^{*(\nu,\mu)}_{V^{(l)}}(z)$ of $U_q(\widehat{\mathfrak{sl}}_2)$, we define the $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ intertwiners $\Phi^{(\nu,\mu)}_{V^{(l)}}(z,\lambda)$ and $\Psi^{*(\nu,\mu)}_{V^{(l)}}(z,\lambda)$ of spin $l/2$ as follows.

$$\Phi^{(\nu,\mu)}_{V^{(l)}}(z,\lambda) := (\mathrm{id} \otimes \pi_{V^{(l)},z}) F(\lambda) \circ \Phi^{(\nu,\mu)}_{V^{(l)}}(z) \ : V(\mu) \longrightarrow V(\nu) \otimes V_z^{(l)},$$
$$\Psi^{*(\nu,\mu)}_{V^{(l)}}(z,\lambda) := \Psi^{*(\nu,\mu)}_{V^{(l)}}(z) \circ (\pi_{V^{(l)},z} \otimes \mathrm{id}) F(\lambda)^{-1} \ : V_z^{(l)} \otimes V(\mu) \longrightarrow V(\nu).$$

Here $V(\mu)$ is the level k highest weight $U_q(\widehat{\mathfrak{sl}}_2)$ module with the highest weight μ and $V_z^{(l)}$ is the $l+1$ dimensional evaluation representation. Furthermore, we can extend them to the spin $l/2$ VO's acting on the $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ modules $\widehat{V}(\mu) := \bigoplus_{n\in\mathbb{Z}} V(\mu) \otimes e^{nQ}$ as follows.

$$\Phi^{(l)(\nu,\mu)}(z) := \Phi^{(\nu,\mu)}_{V^{(l)}}(z,\lambda) \ : \ \widehat{V}(\mu) \to \widehat{V}(\nu) \otimes V_z^{(l)}, \tag{1.8}$$

$$\Psi^{*(l)(\nu,\mu)}(z) := \Psi^{*(\nu,\mu)}_{V^{(l)}}(z,\lambda) e^{h_1 \otimes Q} \ : \ V_z^{(l)} \otimes \widehat{V}(\mu) \to \widehat{V}(\nu). \tag{1.9}$$

Then from the intertwining relations of $\Phi^{(\nu,\mu)}_{V^{(l)}}(z,\lambda)$ and $\Psi^{*(\nu,\mu)}_{V^{(l)}}(z,\lambda)$, we can derive their analogue for the VO's of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ as follows.

$$\Phi^{(l)(\nu,\mu)}(w)\widehat{L}^+(z) = R^+_{V^{(1)}V^{(l)}}(z/w, P+h_1)\widehat{L}^+(z)\Phi^{(l)(\nu,\mu)}(w), \tag{1.10}$$

$$\widehat{L}^+(z)\Psi^{*(l)(\nu,\mu)}(w) = \Psi^{*(l)(\nu,\mu)}(w)\widehat{L}^+(z)R^{*+}_{V^{(1)}V^{(l)}}(z/w, P-h_1^{(1)}-h_1^{(2)}) \tag{1.11}$$

It is also possible to derive the commutation relations of the VO's. Especially, the spin 1/2 type I VO's satisfy

$$\check{R}(z,P+h_1)\Phi^{(\nu,\mu)}(z_1)\Phi^{(\mu,\kappa)}(z_2) = \sum_{\mu'}\Phi^{(\nu,\mu')}(z_2)\Phi^{(\mu',\kappa)}(z_1)W'\left(\begin{matrix}\kappa & \mu \\ \mu' & \nu\end{matrix}\middle| z_1/z_2\right), \tag{1.12}$$

where $\check{R}(z,s) = PR(z,s)$, $R(z,s) = R^+(z,s)\rho(u)/\rho^+(u)$, $Pa\otimes b = b\otimes a$ and $W' = W|_{r\to k+2}$.

2. Free Field Representation

The CFT limit of the ABF model and its fusion models are described by the coset $(\widehat{\mathfrak{sl}}_2)_k \oplus (\widehat{\mathfrak{sl}}_2)_{r-k-2}/(\widehat{\mathfrak{sl}}_2)_{r-2}$ Virasoro minimal model, which is known to be equivalent to the tensor product of the $\mathbb{Z}_k$-parafermion theory and one boson theory with a certain background charge. Corresponding to this fact, the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ can be realized by using the q-deformed $\mathbb{Z}_k$-parafermion theory and one q-boson theory.

2.1. *Drinfeld currents [3]*

We use three kinds of bosons satisfying the relations.

$$[a_{0,n}, a_{0,m}] = \delta_{n+m,0}\frac{[2n][kn]}{n}\frac{[rn]}{[r^*n]}, \quad [P_0,Q_0] = \frac{2kr}{r^*},$$

$$[a_{1,n}, a_{1,m}] = \delta_{n+m,0}\frac{[2n][(k+2)n]}{n}, \qquad [P_1,Q_1] = 2(k+2),$$

$$[a_{2,n}, a_{2,m}] = -\delta_{n+m,0}\frac{[2n][kn]}{n}, \qquad [P_2,Q_2] = -2k.$$

We also set $a'_{0,n} = \frac{[r^*n]}{[rn]}a_{0,n}$. As usual, it is convenient to introduce the corresponding boson fields.

$$\phi_j(A;B,C|z;D) = -\frac{A}{BC}(Q_j + P_j\log z) + \sum_{m\neq 0}\frac{[Am]}{[Bm][Cm]}a_{j,m}z^{-m}q^{D|m|},$$

$$\phi_j^{(\pm)}(A;B|z;C) = \frac{P_j}{2}\log q + (q-q^{-1})\sum_{m>0}\frac{[Am]}{[Bm]}a_{j,\pm m}z^{\mp m}q^{Cm} \quad (j=0,1,2)$$

and $\phi'_0(A;B,C|z) = \phi_0(A;B,C|z)|_{r\to r^*, a_{0,n}\to a'_{0,n}}$. We often use the abridgment $\phi_j(C|z;D) = \phi_j(A;A,C|z;D)$, $\phi_j(C|z) = \phi_j(C|z;0)$ etc.

Now let us define the q-analogue of the $\mathbb{Z}_k$-parafermion fields $\Psi(z)$ and $\Psi^\dagger(z)$ by $\Psi(z) = \Psi^-(z)$, $\Psi^\dagger(z) = \Psi^+(z)$ with

$$\begin{aligned}\Psi^\pm(z) = \mp\frac{1}{(q-q^{-1})} : \exp\left\{\pm\phi_2(c|z;\pm\frac{c}{2})\right\} \\ \times\Big(\exp\left\{-\phi_2^{(+)}\Big(1;2|z;\mp\frac{c+2}{2}\Big) \pm \phi_1^{(+)}\Big(1;2|z;\mp\frac{c}{2}\Big)\right\} \\ -\exp\left\{\phi_2^{(-)}\Big(1;2|z;\mp\frac{c+2}{2}\Big) \mp \phi_1^{(-)}\Big(1;2|z;\mp\frac{c}{2}\Big)\right\}\Big) : .\end{aligned}$$

The $\mathbb{Z}_k$-parafermion theory contains the screening current $S(z)$, which commutes with $\Psi^\pm(z)$ up to the total difference.

$$\begin{aligned}S(z) = \frac{-1}{q-q^{-1}} : \exp\Big\{\phi_1\Big(k+2\Big|z;-\frac{k+2}{2}\Big)\Big\} \\ \times\Big(\exp\Big\{\phi_2^{(+)}\Big(1;2\Big|z;\frac{k+2}{2}\Big) + \phi_1^{(+)}\Big(1;2\Big|z;\frac{k}{2}\Big)\Big\} \\ -\exp\Big\{-\phi_2^{(-)}\Big(1;2\Big|z;\frac{k+2}{2}\Big) - \phi_1^{(-)}\Big(1;2\Big|z;\frac{k}{2}\Big)\Big\}\Big) : .\end{aligned}$$

Then we have,

Proposition 2.1: *The Drinfeld currents of* $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ *at* $c = k$ *are given by*

$$K(u) = z^{-\frac{k}{4rr^*}}\ e^{-\phi_0'(1;2,r^*|z)}, \quad E(u) = \Psi(z)\ e^{-\phi_0(k|z)}, \quad F(u) = \Psi^\dagger(z)\ e^{\phi_0'(k|z)}.$$

We regard these currents as the operators acting on the following Fock spaces.

$$\begin{aligned}\mathcal{F}_{a,m;J} &= \bigoplus_M \mathcal{F}^{PF}_{J,M} \otimes \mathcal{F}^{\phi_0}_{a,m}, \\ \mathcal{F}^{PF}_{J,M} &= \mathbb{C}[a_{1,l}, a_{2,l}\ (l \in \mathbb{Z}_{<0})] \otimes e^{\frac{J}{2(k+2)}Q_1} \otimes e^{\frac{M}{2k}Q_2}, \\ \mathcal{F}^{\phi_0}_{a,m} &= \mathbb{C}[a_{0,l}\ (l \in \mathbb{Z}_{<0})] \otimes e^{\sqrt{\frac{r^*}{kr}}\alpha_{a,m}Q_0},\end{aligned}$$

where $\alpha_{a,m} = \frac{1-a}{2}\alpha_- + \frac{1-m}{2}\alpha_+$, $(1 \le a \le r-1,\ 1 \le m \le r-k-1)$, $\alpha_+ = \sqrt{\frac{r}{kr^*}}$, $\alpha_- = -\sqrt{\frac{r^*}{kr}}$. In fact, $E(z) : \mathcal{F}_{a,m;J} \to \mathcal{F}_{a,m-2;J}$, $F(z) : \mathcal{F}_{a,m;J} \to \mathcal{F}_{a-2,m;J}$, $S(z) : \mathcal{F}_{a,m;J} \to \mathcal{F}_{a,m;J-2}$.

2.2. *The type I vertex operators*

2.2.1. *The spin-1/2 VO's*

Let $\mu_J = (k-J)\Lambda_0 + J\Lambda_1$ be the level k $\widehat{\mathfrak{sl}}_2$ weight. We set $J' = J + \alpha$ $(\alpha = \pm 1)$ and define the components of the spin 1/2 VO as follows.

$$\Phi^{(\mu_{J'},\mu_J)}(u) = g^{(J',J)}(z) \sum_{\varepsilon=\pm 1} \Phi^\alpha_\varepsilon(z) \otimes v_\varepsilon, \tag{2.13}$$

where $\{v_\varepsilon\}_{\varepsilon=\pm}$ is a basis of the two dimensional representation $V^{(1)}$. We realize $\Phi^\alpha_\varepsilon(z)$ as an operator $\Phi^\alpha_\varepsilon(z) : \mathcal{F}_{a,m;J} \to \mathcal{F}_{a-\varepsilon,m;J+\alpha}$.

In the case $\alpha = +$, solving the "intertwining relation" (1.10) with $l = 1$, we obtain $\Phi^+_\varepsilon(z)$, after a certain gauge transformation, as follows.[4]

$$\Phi^+_-(z) = \frac{1}{[K]}\phi_{1,1}(z) : e^{-\phi'_0(1;2,k|z)} :, \tag{2.14}$$

$$\Phi^+_+(z) = F^+(u)\Phi^+_-(z) = \oint_{C_{F,z}} \frac{dz'}{2\pi i z'} \frac{[u-u'+\frac{1}{2}+K]}{[u-u'-\frac{1}{2}]} F(z')\Phi^+_-(z). \tag{2.15}$$

Here we set $K = P + h_1$. $\phi_{1,1}(z)$ is the q-analogue of the spin $1/2$ parafermion primary fields. In general, the sipn $l/2$ ($l = 0, 1, ..., k$) field $\phi_{l,l}(z)$ is given by

$$\phi_{l,l}(z) = : \exp\left\{-\phi_2\left(l;2,k\middle|z;\frac{k}{2}\right) - \phi_1\left(l;2,k+2\middle|z;\frac{k+2}{2}\right)\right\} : . \tag{2.16}$$

The VO's $\Phi^+_\varepsilon(z)$ satisfy the following commutation relation.

$$\sum_{\varepsilon_1,\varepsilon_2} R(z_1/z_2,K)^{\varepsilon'_2\varepsilon'_1}_{\varepsilon_2\varepsilon_1}\Phi^+_{\varepsilon_1}(z_1)\Phi^+_{\varepsilon_2}(z_2) = \rho'(u_1-u_2)\Phi^+_{\varepsilon'_2}(z_2)\Phi^+_{\varepsilon'_1}(z_1).$$

with $\rho'(u) = \rho(u)|_{r\to k+2}$.

On the other hand, the VO's $\Phi^-_\varepsilon(z)$ require an attachment of the $\mathbb{Z}_k$-parafermion screening current $S(z)$, because among $E(z), F(z), S(z)$, only $S(z)$ can decrease the quantum number J. However since the q-deformed $\mathbb{Z}_k$-parafermion theory has no coalgebra structure, we have no definite guiding principle to fix the rule of $S(z)$-attachment. We do this by hand for the spin $1/2$ VO's requiring the commutation relation (1.12) and extend it to the higher spin VO's by fusion procedure. We thus obtain the following realization of the spin $1/2$ VO's.

$$\Phi^-_\varepsilon(z) = \oint_{C_{S,z}} \frac{dw}{2\pi i w}\, \Phi^+_\varepsilon(z)S(w)\frac{[u-v-\frac{1}{2}-P_1]'}{[u-v-\frac{1}{2}]'}. \tag{2.17}$$

Here $[u]' = [u]|_{q^{2r}\to q^{2(k+2)}}$ and the contour $C_{S,z}$ should be chosen in such a way that the poles $z' = q^{k+1}zq^{2(k+2)l}$ ($l = 0, 1, 2, ..$) are inside whereas the poles $z' = q^{-k-1}zq^{-2(k+2)l}$ ($l = 0, 1, 2, ..$) are outside.

2.2.2. *Fusion construction of the higher spin VO's*

We next consider the fusion of the l spin $1/2$ VO's. For $\alpha = \sum_{j=1}^l \alpha_j$, we set

$$\Phi^{(l)\alpha}_\varepsilon(zq^{l-1}) = \sum_{\substack{\varepsilon_1,\dots,\varepsilon_l \\ \sum\varepsilon_i=\varepsilon}} \Phi^{\alpha_1}_{\varepsilon_1}(zq^{2(l-1)})\Phi^{\alpha_2}_{\varepsilon_2}(zq^{2(l-2)})\cdots\Phi^{\alpha_l}_{\varepsilon_l}(z). \tag{2.18}$$

It turns out that the simplest component $\Phi^{(l)l}_{-l}(z)$ is expressed by using the spin $l/2$ q-parafermion primary field $\phi_{l,l}(z)$

$$\begin{aligned}\Phi^{(l)l}_{-l}(zq^{l-1}) &= \Phi^+_-(zq^{2(l-1)})\Phi^+_-(zq^{2(l-2)})\cdots\Phi^+_-(z) \\ &= \left(\frac{1}{\prod_{j=0}^{l-1}[K-j]}\right) C(z)\phi_{l,l}(zq^{l-1}) : e^{-\phi'_0(l;2,k|zq^{l-1})}\end{aligned}$$

with $C(z)$ being a function appearing from the normal ordering.

Lemma 2.2: *In (2.18), let j_a $(1 \leq a \leq m(\leq l))$ be integers satisfying $\alpha_{j_a} = -$, and for other j, $\alpha_j = +$. Then in (2.18) one can move all $S(z)$ in $\Phi^-_{\varepsilon_{j_a}}(z)$ to the right and obtains the expression which has no j_a $(a = 1, .., m)$ dependence at all.*

$$\sum_{\substack{\varepsilon_1,..,\varepsilon_l \\ \sum \varepsilon_i = \varepsilon}} \Phi^+_{\varepsilon_1}(zq^{2(l-1)})\Phi^+_{\varepsilon_2}(zq^{2(l-2)}) \cdots \Phi^+_{\varepsilon_l}(z) \left(\oint_{C_{S,z}} \underline{dw} S(w) \frac{[u - v - \frac{1}{2} - P_1]'}{[u - v - \frac{1}{2}]'} \right)^m .$$

This indicates that the VO's $\Phi^{(l)\alpha}_{\varepsilon}(z)$ have no intermediate-weight-path dependence. Hence $\Phi^{(l)\alpha}_{\varepsilon}(z)$ manifestly satisfies the commutation relation for the spin $l/2$ VO's.

$$\sum_{\varepsilon_1,\varepsilon_2} R_{ll}(z_1/z_2, K)^{\varepsilon_2'\varepsilon_1'}_{\varepsilon_2\varepsilon_1} \Phi^{(l)\alpha_1}_{\varepsilon_1}(z_1)\Phi^{(l)\alpha_2}_{\varepsilon_2}(z_2)$$

$$= \sum_{\alpha_1',\alpha_2'} W'_{ll} \left(\begin{matrix} \Pi & \Pi - \alpha_1 \\ \Pi - \alpha_2' & \Pi - \alpha_1' - \alpha_2' \end{matrix} \middle| z_1/z_2 \right) \Phi^{(l)\alpha_2'}_{\varepsilon_2'}(z_2)\Phi^{(l)\alpha_1'}_{\varepsilon_1'}(z_1),$$

where $\Pi = P_1 + 1$, R_{ll} and W'_{ll} are the $l \times l$ fused ABF Boltzmann weights.

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QUANTUM DYNAMICS AND RANDOM MATRIX THEORY

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We compute the survival probability of an initial state, with an energy in a certain window, by means of random matrix theory. We determine its probability distribution and show that is is universal, i.e. characterised only by the symmetry class of the hamiltonian and independent of the initial state.

In classical mechanics, temporal chaos is characterised by the extreme sensibility of a trajectory to variation of initial conditions. No direct analog of this phenomenon has been found in quantum mechanics so far. On the other hand, numerical evidence has been accumulated,[1] showing that energy levels of a quantum system, whose classical counterpart is chaotic, have a statistical behavior described by Wigner's random matrix theory (RMT), on the mean level spacing scale. The question we want to address is the following: are there specific predictions of RMT for quantum dynamics, which would characterise the temporal behavior of "chaotic" quantum systems.

We consider the following situation: The system is prepared in an initial state φ at time 0, with an energy in a certain window, centered at e and of width $2sl(e)$, where $l(e)$ is the mean level spacing, and we want to compute the probability to find our system again in the state φ, at a later time t. This quantity that we call the *survival probability* R is given by

$$R = \left| \frac{(\varphi | \exp i\frac{t}{\hbar} H \; P(\Delta)\varphi)}{(\varphi,\, P(\Delta)\varphi)} \right|^2 \tag{1}$$

H is the hamiltonian of our system and $P(\Delta)$ is the spectral projector on Δ.

We have chosen to take an energy in a range of the order of the mean level spacing in order to look at properties of the system which are independent of specific details. If

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(λ_j, ψ_j) denote respectively the j^th eigenvalue and eigenvector of the Hamiltonian H, then the survival probability can be written as

$$R = \left| \frac{\sum_{j=1} y_j \chi_j \exp 2\pi i \tau x_j}{\sum_{j=1} y_j \chi_j} \right|^2 \Theta \left(\sum_j \chi_j - 1 \right) \tag{2}$$

where

$$y_j = |(\varphi, \psi_j)|^2 \tag{3}$$

and if we define x_j by the relation

$$\lambda_j = e + x_j l(e) \tag{4}$$

$$\chi_j \equiv \chi_{(-s,s)}(x_j) = \begin{cases} 1 & \text{if } |x_j| \leq s \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

The Heaviside function Θ ensures that there is at least one eigenvalue in Δ. What appears naturally in this expression is the time measured in units of the Heisenberg time

$$t_H = \frac{h}{l(e)} \tag{6}$$

so that

$$\tau = \frac{t}{t_H}. \tag{7}$$

If we look at this problem from the point of view of RMT, we will replace the Hamiltonian by a large $N \times N$ self-adjoint matrix, whose probability distribution is basis independent and therefore of the form

$$e^{-W(\lambda_1, \ldots, \lambda_N)}\, dH \tag{8}$$

Wigner's gaussian model corresponds to the choice

$$W = \frac{N}{2} \sum_{j=1}^{N} \lambda_j^2 \tag{9}$$

The first conclusion to be drawn is that the survival probability is statistically *independent of the initial state* φ. This follows from the fact that the variables $\{y_j\}_{j=1}^N$ have a probability distribution, independent of φ and given by:

$$\mu_N(\mathbf{y}) d\mathbf{y} = \frac{1}{C_N} \delta \left(\sum_{j=1}^{N} y_j - 1 \right) \prod_{j=1}^{N} y_j^{\frac{\beta}{2} - 1}\, d\mathbf{y} \tag{10}$$

The parameter $\beta = 1, 2, 4$ characterise the symmetry class of the Hamiltonian, respectively orthogonal, unitary and symplectic. Equation (10) follows easily from the Haar measure on the corresponding groups. C_N is a normalising constant.

The variables $\{x_j\}_{j=1}^N$ are statistically independent of the variables $\{y_j\}_{j=1}^N$ and have a distribution given by

$$\frac{1}{D_N} \exp -W\left(e + \mathbf{x}l(e)\right)\Delta^\beta\left(\mathbf{x}\right) d\mathbf{x} \tag{11}$$

where the Van der Monde determinant

$$\Delta\left(\mathbf{x}\right) = \prod_{1\leq i<j\leq N} |x_i - x_j| \tag{12}$$

comes from the change of variables $H_{ij} \to (\lambda_j, \psi_j)_{j=1}^N$.[2] D_N is a constant of normalisation.

We can take $l(e) = \frac{1}{N\rho(e)}$, where $\rho(e)$ is the density of states when $N = \infty$.
The problem that we need to solve now is to find the probability distribution of the survival probability $p(R)dR$ in the $N = \infty$ limit. We find that R is **not** *self-averaging* i.e. $p(R)$ is not a delta distribution concentrated on the mean value of R. On the other hand its probability distribution $p(R)$ is *universal*, i.e. it depends only on the symmetry parameter β, at least for a large class of W.
There are two formulas for $p(R)$, one more appropriate to small windows, another one to large windows.
In the first case, we decompose $p(R)$ into

$$p\left(R|\tau\right) = \sum_{n=1}^{\infty} \frac{E_n}{1 - E_0} p_n\left(R|\tau\right) \tag{13}$$

where E_n is the probability to find exactly n eigenvalues in Δ and $p_n(R|\tau)$ is the conditional probability density of R knowing that there are exactly n eigenvalues in Δ.
It can be expressed as

$$p_n(R|\tau) = \int_{-s}^{s} |!\hat{E}(x_1, \ldots, x_n) d^n x \int_0^\infty \mu_n(z_1, \ldots, z_n) d^n z \delta\left(-\left|\sum_{j=1}^{n} z_j \exp 2\pi i \tau x_j\right|^2\right) \tag{14}$$

$$\hat{E}\left(x_1, \ldots, x_n\right) = \frac{E\left(x_1, \ldots, x_n\right)}{E_n} \tag{15}$$

$$E_n = \int_{-s}^{s} E\left(x_1, \ldots, x_n\right) d^n x \tag{16}$$

$E\left(x_1, \ldots, x_n\right)$ being the probability density of finding the n eigenvalues in Δ at $(x_1, \ldots, x_n)$.
Useful expressions for $E\left(x_1, \ldots, x_n\right)$ and E_n can be found in Refs. 2 and 3. It is expressible in terms of a determinant

$$E\left(x_1, \ldots, x_n\right) = \det L_\beta\left(x_i | x_j\right) \quad ; \quad (i, j) \in (1 \ldots n) \tag{17}$$

where

$$L_\beta = \frac{K_\beta}{1 - K_\beta} \tag{18}$$

K_β is an operator whose kernel in the simplest case $\beta = 2$ is given by

$$K_\beta(x|y) = \frac{\sin \pi(x-y)}{\pi(x-y)} \tag{19}$$

defined on $L^2(-s, s)$.

Universality comes from the fact that $E(x_1, \ldots, x_n)$ is expressible in terms of the correlation functions and the latter ones depends only on β, for a large class of W. W modifies only the density of states and therefore the mean level spacing $l(e)$. This expression for $p(R|\tau)$ is mostly useful in the small window limit, because when $s \to 0$

$$E_n \sim s^{\frac{\beta}{2}n^2 + n(1-\frac{\beta}{2})} \tag{20}$$

Moreover in this case we have

$$\lim_{s \to 0} s^n \hat{E}(sx_1, \ldots, sx_n) = A_n \prod_{1 \le i < j \le n} |x_i - x_j|^\beta \tag{21}$$

so that the probability distribution of R shows a *scaling behavior*

$$\lim_{\substack{s \to 0 \\ R \to 1}} s^{-\beta-1} pr\left\{\frac{1-R}{(\pi\tau s)^2} \ge x\right\} = \int_x^1 g_\beta(\lambda) d\lambda \tag{22}$$

the function $g_\beta(\lambda)$ being given by

$$g_\beta(\lambda) = A_\beta \lambda^{\frac{\beta-1}{2}} \left[\frac{1}{2}\sqrt{1-\lambda} + \frac{1}{2}\ln\lambda - \ln 1 + \sqrt{1-\lambda}\right] \tag{23}$$

On the other hand, one can see from eq (13) and (14) that the probability distribution of R is well defined at *infinite times.* Namely

$$p_n(R|\tau) = p_n(R|\infty) + O\left(\frac{1}{\tau}\right) \tag{24}$$

as can be seen by an integration by parts where

$$p_n(R|\infty) = \int_0^\infty \mu_n(z_1, \ldots, z_n) \int_0^{2\pi} \prod_{j=1}^n \frac{d\phi_j}{2\pi} \delta\left(R - \left|\sum_{j=1}^n z_j e^{i\phi_j}\right|^2\right) \tag{25}$$

Using an integral representation for the delta appearing in the definition (10) of the μ_n, we can reexpress (25) as

$$p_n(R|\infty) = \frac{1}{4\pi c_n} \int_{\epsilon - i\infty}^{\epsilon + i\infty} du\, e^u \int_0^{+\infty} dr\, r J_0\left(\sqrt{R}r\right) \left[\int_0^\infty dz\, e^{-uz} J_0(rz)\, z^{\frac{\beta}{2}-1}\right]^n \tag{26}$$

ϵ being any positive number, and $J_0(x)$ the Bessel function. This expression can be simplified, considerably when $\beta = 2, 4$.

In the unitary case ($\beta = 2$) one finds

$$p_n(R|\infty) = \frac{n-1}{2}(1-R)^{\frac{n-3}{2}} \tag{27}$$

For a *large window* of energy, it is more appreciate to find another expression for $p_n(R|\tau)$. It is given as some integral over a Fredholm determinant $\mathcal{G}$.
$\mathcal{G}$ is a generating function for the variables $\{y_j\}$ and $\{x_j\}$ appearing in the definition of R, eq (2).

$$\mathcal{G}(r;\varphi;z) = \lim_{N\to\infty} \left\langle \exp -iN \sum_{j=1}^{N} y_j \chi_j \left[r\cos(2\pi\tau x_j + \varphi) + z \right] \right\rangle \tag{28}$$

It can be expressed in terms of the operator K_β appearing in eq (19), when $\beta = 1, 2$ as

$$\mathcal{G} = E_0 \left[\det \left(1 + K_\beta g^{\frac{\beta}{2}} \right) \right]^{\frac{\beta}{2}} \tag{29}$$

with

$$E_0 = \left[\det \left(1 - K_\beta \right) \right]^{\frac{\beta}{2}} \tag{30}$$

and g is the multiplication operator by the function

$$g = \left[1 + \frac{2i}{\beta} \left[z + r\cos(2\pi\tau x + \varphi) \right] \right]^{-1} \tag{31}$$

When the window is large ($s >> 1$) we can expand the determinant in powers of K_β, the first two terms of this expansion dominating the other ones.[4]
One finds that the probability distribution is *exponential.*

$$\lim_{s\to\infty} \frac{1}{s} p\left(\frac{R}{s} \middle| \tau \right) = \frac{1}{\sigma(\tau)} \exp -\frac{R}{\sigma(\tau)} \tag{32}$$

In the *orthogonal case* ($\beta = 1$), for example

$$\sigma(\tau) = \begin{cases} 4 - 2|\tau| + |\tau| \ln 1 + 2|\tau| & \text{if } |\tau| \le 1 \\ 2 + |\tau| \ln \frac{2|\tau|+1}{2|\tau|-1} & \text{if } |\tau| \ge 1 \end{cases} \tag{33}$$

One can notice the *singularity at the Heisenberg time* $\tau = 1$ and the fact that $\sigma(\infty)$ exists.
However if we *smooth out* in time $R(\tau)$, taking for example

$$\overline{R} = \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} R(\tau) d\tau \tag{34}$$

then we get a *self-averaging quantity*

$$\lim_{s\to\infty} \frac{1}{s} p\left(\frac{\overline{R}}{s} \right) = \delta\left(\overline{R} - \overline{\sigma} \right) \tag{35}$$

with

$$\sigma = \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} d\tau\, \sigma(\tau) \tag{36}$$

Some numerical work on chaotic billiards,[5] in the large window limit, confirm this exponential distribution. Integrable billiards show a very different behaviour.[5]

Finally, we would like to mention the fact that Wigner's energy level statistics can be obtained for models, where eigenvalues and eigenvectors are correlated. We think therefore that the study of quantum dynamics could discriminate between such models and those we have considered where they are uncorrelated.

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INTEGRABLE COUPLING IN A MODEL FOR JOSEPHSON TUNNELING BETWEEN NON-IDENTICAL BCS SYSTEMS

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We extend a recent construction for an integrable model describing Josephson tunneling between identical BCS systems to the case where the BCS systems have different single particle energy levels. The exact solution of this generalized model is obtained through the Bethe ansatz.

1. Introduction.

The experimental work of Ralph, Black and Tinkham[1,2] on the discrete energy spectrum in small metallic aluminium grains has generated substantial interest in understanding the nature of superconducting correlations at the nano-scale level. Their results indicate significant parity effects due to the number of electrons in the system. For grains with an odd number of electrons, the gap in the energy spectrum reduces with the size of the system, in contrast to the case of a grain with an even number of electrons, where a gap larger than the single electron energy levels persists. In the latter case the gap can be closed by a strong applied magnetic field. The conclusion drawn from these results is that pairing interactions are prominent in these nano-scale systems. For a grain with an odd number of electrons there will always be at least one unpaired electron, so it is not necessary to break a Cooper pair in order to create an excited state. For a grain with an even number of electrons, all excited states have a least one broken Cooper pair, resulting in a gap in the spectrum. In the presence of a strongly applied magnetic field, it is energetically more favourable for a grain with an even number of electrons to have broken pairs, and hence in this case there are excitations which show no gap in the spectrum.

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The physical properties of a small metallic grain are described by the reduced BCS Hamiltonian[3,4]

$$H_{BCS} = \sum_{j=1}^{\mathcal{L}} \epsilon_j n_j - g \sum_{j,k}^{\mathcal{L}} c_{k+}^{\dagger} c_{k-}^{\dagger} c_{j-} c_{j+}. \tag{1}$$

Above, $j = 1, ..., \mathcal{L}$ labels a shell of doubly degenerate single particle energy levels with energies ϵ_j and n_j is the fermion number operator for level j. The operators $c_{j\pm}$, $c_{j\pm}^{\dagger}$ are the annihilation and creation operators for the fermions at level j. The labels $\pm$ refer to time reversed states.

One of the prominent features of the Hamiltonian (1) is the *blocking effect.* For any unpaired electron at level j the action of the pairing interaction is zero since only paired electrons are scattered. This means that the Hilbert space can be decoupled into a product of paired and unpaired electron states in which the action of the Hamiltonian on the subspace for the unpaired electrons is automatically diagonal in the natural basis. In view of the blocking effect, it is convenient to introduce hard-core boson operators $b_j = c_{j-} c_{j+}$, $b_j^{\dagger} = c_{j+}^{\dagger} c_{j-}^{\dagger}$ which satisfy the relations

$$(b_j^{\dagger})^2 = 0, \quad [b_j, b_k^{\dagger}] = \delta_{jk}(1 - 2b_j^{\dagger} b_j)$$

$$[b_j, b_k] = [b_j^{\dagger}, b_k^{\dagger}] = 0$$

on the subspace excluding single particle states. In this setting the hard-core boson operators realise the $su(2)$ algebra in the pseudo-spin representation, which will be utilized below.

The original approach of Bardeen, Cooper and Schrieffer[5] to describe the phenomenon of superconductivity was to employ a mean field theory using a variational wavefunction for the ground state which has an undetermined number of electrons. The expectation value for the number operator is then fixed by means of a chemical potential term μ. One of the predictions of the BCS theory is that the number of Cooper pairs in the ground state of the system is given by the ratio Δ/d where Δ is the BCS "bulk gap" and d is the mean level spacing for the single electron eigenstates. For nano-scale systems, this ratio is of the order of unity, in seeming contradiction with the experimental results discussed above. The explanation for this is that the mean-field approach is inappropriate for nano-scale systems due to large superconducting fluctuations.

As an alternative to the BCS mean field approach, one can appeal to the exact solution of the Hamiltonian (1) derived by Richardson and Sherman.[6] It has also been shown by Cambiaggio, Rivas and Saraceno[7] that (1) is integrable in the sense that there exists a set of mutually commutative operators which commute with the Hamiltonian. These features have recently been shown to be a consequence of the fact that the model can be derived in the context of the Quantum Inverse Scattering Method (QISM) using a solution of the Yang-Baxter equation associated with the Lie algebra $su(2)$.[8,9] One of the aims of the present work is to extend this approach

for application to generalised models. As a specific example, we will show that a model for strong Josephson coupling between two BCS systems falls into this class.

Recall first that electron pairing interactions manifest themselves in macroscopic systems via three well known phenomena:

- supercurrents
- Meissner effect
- Josephson effect

As noted by von Delft,[4] the notion of a supercurrent in a nano-scale system is inapplicable because the mean fee path of an electron is comparable to the system size. Likewise, the penetration depth of an applied magnetic field is comparable to the system size, which prohibits any Meissner effect.

Josephson[11] put forth a proposal for the tunneling of electron pairs between superconductors separated by an insulating barrier. A theory was derived to describe *weak* coupling between two superconductors treated at the mean field level in the grand-canonical ensemble. A remarkable prediction of the theory was that it is possible for a direct current to flow across the insulator for the case of zero applied voltage, whereas a constant voltage across the insulator produces an alternating current. The essential features of the theory stem from the phase difference between the superconductors, which is well defined since the variational wavefunctions for the superconductors have undetermined particle numbers.

For the case of nano-scale systems, the above predictions are again invalid due to the finite particle numbers for each system, giving rise to phase uncertainty. However, if we are to consider *strong* coupling where individual particle numbers are not conserved, only total particle number, it is appropriate to study the effective Hamiltonian

$$H = H_{BCS}(1) + H_{BCS}(2) - \varepsilon_J \sum_{j,k}^{\mathcal{L}} \left(b_j^\dagger(1) b_k(2) + b_j^\dagger(2) b_k(1) \right), \tag{2}$$

where ε_J is the Josephson coupling energy, for the purpose of investigating the nature of pair tunneling at the nano-scale level. In a previous work[10] it was shown that the above Hamiltonian is integrable for $\varepsilon_J = g$ for the case when $H_{BCS}(1)$, $H_{BCS}(2)$ have identical single electron energy levels. Below we will extend this construction to the case where $H_{BCS}(1)$, $H_{BCS}(2)$ describe non-identical systems.

2. A universal integrable system.

First we introduce the Lie algebra $su(2)$ with generators S^+, S^-, S^z satisfying the commutation relations

$$[S^z, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^z. \tag{3}$$

The Casimir invariant, which commutes with each element of the algebra, has the form

$$C = S^+S^- + S^-S^+ + 2\left(S^z\right)^2.$$

Associated with the $su(2)$ algebra there is a solution of the Yang-Baxter equation in End$V \otimes$ End$V \otimes su(2)$, where V denotes a two-dimensional vector space. This solution reads[12]

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v)$$

with

$$R(u) = I \otimes I + \frac{\eta}{u}\sum_{m,n}^{2} e_n^m \otimes e_m^n,$$

$$L(u) = I \otimes I + \frac{\eta}{u}\left(e_1^1 \otimes S^z - e_2^2 \otimes S^z + e_2^1 \otimes S^- + e_1^2 \otimes S^+\right)$$

where $\{e_n^m\}$ are 2×2 matrices with 1 in the (m,n) entry and zeroes elsewhere. Above, I is the identity operator and η is a scaling parameter for the rapidity variable u which plays an important role in the subsequent analysis. With this solution we construct the transfer matrix

$$t(u) = \mathrm{tr}_0\left(G_0 L_{0\mathcal{L}}(u-\epsilon_{\mathcal{L}})...L_{01}(u-\epsilon_1)\right) \tag{4}$$

which is an element of the $\mathcal{L}$-fold tensor algebra of $su(2)$. Above, tr_0 denotes the trace taken over the auxiliary space and $G = \exp(\alpha\eta\sigma)$ with $\sigma = \mathrm{diag}(1,-1)$. A consequence of the Yang-Baxter equation is that $[t(u),\, t(v)] = 0$ for all values of the parameters u and v, and independent of the representations of $su(2)$ in the tensor algebra. Defining

$$T_j = \lim_{u\to\epsilon_j} \frac{u-\epsilon_j}{\eta^2} t(u)$$

for $j = 1, 2, ..., \mathcal{L}$, we may write in the *quasi-classical limit* $T_j = \tau_j + o(\eta)$ and it follows that $[\tau_j,\, \tau_k] = 0,\ \forall\, j,\, k$. Explicitly, these operators read

$$\tau_j = 2\alpha S_j^z + \sum_{k\neq j}^{\mathcal{L}} \frac{\theta_{jk}}{\epsilon_j - \epsilon_k} \tag{5}$$

with $\theta = S^+ \otimes S^- + S^- \otimes S^+ + 2S^z \otimes S^z$.

We define a Hamiltonian through

$$H = -\frac{1}{\alpha}\sum_{j}^{\mathcal{L}} \epsilon_j\tau_j + \frac{1}{4\alpha^3}\sum_{j,k}^{\mathcal{L}} \tau_j\tau_k + \frac{1}{2\alpha^2}\sum_{j}^{\mathcal{L}} \tau_j - \frac{1}{2\alpha}\sum_{j}^{\mathcal{L}} C_j \tag{6}$$

$$= -\sum_{j}^{\mathcal{L}} 2\epsilon_j S_j^z - \frac{1}{\alpha}\sum_{j,k}^{\mathcal{L}} S_j^- S_k^+. \tag{7}$$

The Hamiltonian is *universally* integrable since it is clear that $[H,\, \tau_j] = 0,\quad \forall j$ irrespective of the realizations of the $su(2)$ algebra in the tensor algebra.

Realizing the $su(2)$ generators through the hard-core bosons; viz

$$S_j^+ = b_j, \quad S_j^- = b_j^\dagger, \quad S_j^z = \frac{1}{2}(I - n_j) \tag{8}$$

one obtains (1) (up to a constant) with $g = 1/\alpha$ as shown by Zhou et al.[8] and von Delft and Poghossian.[9]

We now turn to applying (7) for the study of two coupled BCS systems. To accommodate this, it is convenient to first consider three index sets P_0, P_1, P_2 such that individually the BCS Hamiltonians are expressible

$$H_{BCS}(i) = \sum_{j \in (P_0 \cup P_i)}^{\mathcal{L}} \epsilon_j n_j - g \sum_{j,k \in (P_0 \cup P_i)}^{\mathcal{L}} b_k^\dagger b_j.$$

If the single particle energy ϵ_j is common to both systems, then $j \in P_0$. Hence it is meant to be understood that $\epsilon_j \neq \epsilon_k \neq \epsilon_l \, \forall \, j \in P_1$, $k \in P_2$, $l \in P_3$. In the case that $j \in P_0$, the local $su(2)$ operators are described by the tensor product of two pseudo-spin realisations acting on the four-dimensional tensor product space. We can now realise (7) in terms of the hard-core boson representation (8)

$$S_j^+ = b_j(i), \quad S_j^- = b_j^\dagger(i), \quad S_j^z = \frac{1}{2}(I - n_j(i))$$

for $j \in P_i$, $i = 1, 2$ whereas for $j \in P_0$ we take the tensor product representation

$$\begin{aligned} S_j^+ &= b_j(1) + b_j(2) \\ S_j^- &= b_j^\dagger(1) + b_j^\dagger(2) \\ S_j^z &= I - \frac{1}{2}\left(n_j(1) + n_j(2)\right). \end{aligned}$$

Under this representation of (7) we obtain (2) with $\varepsilon_J = g = 1/\alpha$, establishing integrability at this value of the Josephson coupling energy. For the case when the index sets P_1, P_2 are both empty, i.e., the two BCS systems are identical, this result was previously shown by Links et al..[10]

3. The exact solution.

In addition to proving integrability for $\varepsilon_J = g$, we can also obtain the exact solution from the Bethe ansatz. Below we will derive the energy eigenvalues for the Hamiltonian (7) in a very general context, which includes those of (2) with $\varepsilon_J = g$ as a particular case.

For each index k in the tensor algebra in which the transfer matrix acts, and accordingly in (7), suppose that we represent the $su(2)$ algebra through the irreducible representation with spin s_k. Thus $\{S_k^+, S_k^-, S_k^z\}$ act on a $(2s_k + 1)$-dimensional space. Employing the standard method of the algebraic Bethe ansatz[12] gives that the eigenvalues of the transfer matrix (4) take the form

$$\Lambda(u) = \exp(\alpha\eta) \prod_k^{\mathcal{L}} \frac{u - \epsilon_k + \eta s_k}{u - \epsilon_k} \prod_j^M \frac{u - w_j - \eta}{u - w_j}$$

$$+\exp(-\alpha\eta)\prod_{k}^{\mathcal{L}}\frac{u-\epsilon_k-\eta s_k}{u-\epsilon_k}\prod_{j}^{M}\frac{u-w_j+\eta}{u-w_j}.$$

Above, the parameters w_j are required to satisfy the Bethe ansatz equations

$$\exp(2\alpha\eta)\prod_{k}^{\mathcal{L}}\frac{w_l-\epsilon_k+\eta s_k}{w_l-\epsilon_k-\eta s_k}=-\prod_{j}^{M}\frac{w_l-w_j+\eta}{w_l-w_j-\eta}.$$

The eigenvalues of the conserved operators (5) are obtained through the appropriate terms in the expansion of the transfer matrix eigenvalues in the parameter η. This yields the following result for the eigenvalues λ_j of τ_j

$$\lambda_j=\left(2\alpha+\sum_{k\neq j}^{\mathcal{L}}\frac{2s_k}{\epsilon_j-\epsilon_k}-\sum_{i}^{M}\frac{2}{\epsilon_j-v_i}\right)s_j \tag{9}$$

such that the parameters v_j satisfy the coupled algebraic equations

$$2\alpha+\sum_{k}^{\mathcal{L}}\frac{2s_k}{v_j-\epsilon_k}=\sum_{i\neq j}^{M}\frac{2}{v_j-v_i}. \tag{10}$$

Through (9) we can now determine the energy eigenvalues of (7). It is useful to note the following identities

$$2\alpha\sum_{j}^{M}v_j+2\sum_{j}^{M}\sum_{k}^{\mathcal{L}}\frac{v_j s_k}{v_j-\epsilon_k}=M(M-1)$$

$$\alpha M+\sum_{j}^{M}\sum_{k}^{\mathcal{L}}\frac{s_k}{v_j-\epsilon_k}=0$$

$$\sum_{j}^{M}\sum_{k}^{\mathcal{L}}\frac{v_j s_k}{v_j-\epsilon_k}-\sum_{j}^{M}\sum_{k}^{\mathcal{L}}\frac{s_k\epsilon_k}{v_j-\epsilon_k}=M\sum_{k}^{\mathcal{L}}s_k.$$

Employing the above it is deduced that

$$\sum_{j}^{\mathcal{L}}\lambda_j=2\alpha\sum_{j}^{\mathcal{L}}s_j-2\alpha M$$

$$\sum_{j}^{\mathcal{L}}\epsilon_j\lambda_j=2\alpha\sum_{j}^{\mathcal{L}}\epsilon_j s_j+\sum_{j}^{M}\sum_{k\neq j}^{M}s_j s_k-2M\sum_{k}^{\mathcal{L}}s_k-2\alpha\sum_{j}^{M}v_j+M(M-1)$$

which, combined with the eigenvalues $2s_j(s_j+1)$ for the Casimir invariants C_j, yields the energy eigenvalues

$$E=2\sum_{j}^{M}v_j-2\sum_{k}^{\mathcal{L}}s_k\epsilon_k. \tag{11}$$

From the above expression we see that the quasi-particle excitation energies are given by twice the Bethe ansatz roots $\{v_j\}$ of (10).

In order to specialise this result to (2) at integrable coupling, it is useful to first make the following observation. For $j \in P_0$, in which case the $su(2)$ algebra is realised via the tensor product of two hard-core boson representations, it is well known that the representation space is completely reducible into triplet states and a singlet state. Note however, that for the singlet state the $su(2)$ generators act trivially, and hence this state is blocked from scattering in analogy with the blocking of single particle states discussed in the introduction. Hence the $su(2)$ algebra will only act non-trivially on the triplet states. In specialising (10,11) to the case of (2), we need only to set $s_j = 1/2$ for $j \in P_1 \cup P_2$ and $s_j = 1$ for $j \in P_0$.

4. Conclusion

We have displayed the existence of a general class of integrable systems which includes the reduced BCS Hamiltonian and a model for strong Josephson tunneling between two reduced BCS systems. By deriving the models through the QISM we have also determined the exact solution via the Bethe ansatz. A further application of this approach is the computation of form factors and correlation functions.[8,10]

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FINITE DENSITY ALGORITHM IN LATTICE QCD—A CANONICAL ENSEMBLE APPROACH

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I will review the finite density algorithm for lattice QCD based on finite chemical potential and summarize the associated difficulties. I will propose a canonical ensemble approach which projects out the finite baryon number sector from the fermion determinant. For this algorithm to work, it requires an efficient method for calculating the fermion determinant and a Monte Carlo algorithm which accommodates unbiased estimate of the probability. I shall report on the progress made along this direction with the Padé–Z_2 estimator of the determinant and its implementation in the newly developed Noisy Monte Carlo algorithm.

1. Introduction

Fermions at finite density or finite chemical potential is a subject of a wide range of interest. It is relevant to condensed matter physics, such as the Hubbard model away from half-filling. The research about nuclei and neutron stars at low and high nucleon density is actively pursued in nuclear physics and astrophysics. The subject of quark gluon plasma is important for understanding the early universe and is being sought for in relativistic heavy-ion collisions in the laboratories. Furthermore, speculation about color superconducting phase has been proposed recently for quantum chromodynamics (QCD) at very high quark density.[1]

Although there are models, e.g. chiral models and the Nambu–Jona-Lasinio model which have been used to study QCD at finite quark density, the only way to study QCD at finite density and temperature reliably and systematically is via lattice gauge calculations. There have been extensive lattice calculations of QCD at finite temperature.[2] On the contrary, the calculation at finite density is hampered by the lack of a viable algorithm.

In this talk, I shall first review the difficulties associated with the finite density algorithm with chemical potentials in Sec. 2. I will then outline in Sec. 3 a proposal for a finite density algorithm in the canonical ensemble which projects out the

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nonzero baryon number sector from the fermion determinant. In Sec. 4, a newly developed Noisy Monte Carlo algorithm which admits unbiased estimate of the probability is described. Its application to the fermion determinant is outlined in Sec. 5. I will discuss an efficient way, the Padé–Z_2 method, to estimate the Tr log of the fermion matrix in Sec. 6. The recent progress on the implementation of the Kentucky Noisy Monte Carlo algorithm to dynamical fermions is presented in Sec. 7. Finally, a summary is given in Sec. 8.

2. Finite Chemical Potential

The usual approach to the finite density in the Euclidean path-integral formalism of lattice QCD is to consider the grand canonical ensemble with the partition function

$$Z_{GC}(\mu) = \sum_N Z_N e^{-\mu N} = \int \mathcal{D}U \det M[U,\mu] e^{-S_g[U]}, \tag{1}$$

where the fermion fields with fermion matrix M has been integrated to give the determinant. U is the gauge link variable and S_g is the gauge action. The chemical potential is introduced to the quark action with the $e^{\mu a}$ factor in the time-forward hopping term and $e^{-\mu a}$ in the time-backward hopping term. Here a is the lattice spacing. However, this causes the fermion action to be non-Hermitian, i.e. $\gamma_5 M \gamma_5 \neq M$. As a result, the fermion determinant $\det M[U]$ is complex and this leads to the infamous sign problem.

There are several approaches to avoid the sign problem:

2.1. *Fugacity Expansion*

It was proposed by the Glasgow group[3] that the sign problem can be circumvented based on the expansion of the grand canonical partition function in powers of the fugacity variable $e^{\mu/T}$,

$$Z_{GC}(\mu/T, T, V) = \sum_{B=-3V}^{B=3V} e^{\mu/T\, B} Z_B(T,V), \tag{2}$$

where Z_B is the canonical partition function for the baryon sector with baryon number B. Z_{GC} is calculated with reweighting of the fermion determinant

$$Z_{GC}(\mu) = \langle \frac{\det M[U,\mu]}{\det M[U,0]} \rangle_{\mu=0}. \tag{3}$$

Since the reweighting is based on the gauge configuration with $\mu = 0$, it avoids the sign problem. However, this does not work, except perhaps at small μ or near the finite temperature phase transition. We will dwell on this later in Sec. 3. This is caused by the 'overlap problem'[4] where the important samples of configurations in the $\mu = 0$ simulation has exponentially small overlap with those relevant for the finite density. As a result, the onset of baryon begins at $\mu \sim m_\pi/2$ instead of the expected $M_N/3$ which resembles the situation of the quenched approximation.

2.2. *Imaginary Chemical Potential*

In this approach, the chemical potential is taking an imaginary value $\mu = i\nu$. The fermion determinant is real in this case and one can avoid the sign problem.[5–7] The partition function is

$$Z_{GC}(i\nu/T, T, V) = \mathrm{Tr}\, e^{-\hat{H}/T} e^{i\nu\hat{B}/T}, \tag{4}$$

which is periodic with respect to ν with a period of $2\pi T$. Comparing with Eq. (2), one can in principle obtain canonical partition function Z_B from the Fourier transform

$$Z_B(T, V) = \frac{1}{2\pi T} \int_0^{2\pi T} d\nu Z_{GC}(i\nu/T, T, V) e^{-i\nu\hat{B}/T}. \tag{5}$$

In this approach, one needs to integrate over the whole range of ν from 0 to $2\pi T$ after one obtains the Monte Carlo configurations of $Z_{GC}(i\nu/T, T, V)$ at different ν. In practice, it is proposed to calculate the following ratio in the two-dimensional Hubbard model,[7]

$$\frac{Z_{GC}(i\nu/T, T, V)}{Z_{GC}(i\nu_0/T, T, V)} = \int \mathcal{D}\phi e^{-S_{bos}} \det M(i\nu_0) \frac{\det M(i\nu)}{\det M(i\nu_0)}, \tag{6}$$

with a reference value ν_0. Several patches each centered around a different reference point ν_0 are used to cover the range of ν. This was successful for the two-dimensional Hubbard model with a $4^2 \times 10$ lattice up to B = 6 where the determinant was calculated exactly. While this works for a small lattice in the Hubbard model, it would not work for reasonably large lattices in QCD. This is because the direct calculation of the determinant is a V^3 (or V^2 for a sparse matrix) operation which is an impracticable task for the quark matrix which is typically of the dimension $10^6 \times 10^6$. Any stochastic estimation of the determinant will inevitably introduce systematic error. Furthermore, this will also suffer from the 'overlap' problem discussed above. Any Monte Carlo simulation at a reference point ν_0 will have exponentially small overlap with those configurations important to a nonzero baryon density.

2.3. *Overlap Ensuring Multi-parameter Reweighting*

To alleviate the sign problem with the real chemical potential and the overlap problem due to reweighting, it is proposed[8] to do the reweighting in the multiple parameter space. The generic partition function Z_{GC} in Eq. (1) is parametrized by a set of parameters α, such as the chemical potential μ, the gauge coupling β, the quark mass m_q, etc. The partition function can be written to facilitate reweighting

$$Z_{GC}(\alpha) = \int \mathcal{D}U \det M[U, \alpha_0] e^{-S_g[U,\alpha_0]} \{ e^{-S_g[U,\alpha]+S_g[U,\alpha_0]} \frac{\det M[U, \alpha]}{\det M[U, \alpha_0]} \}, \tag{7}$$

where the Monte Carlo simulation is carried out with the α_0 set of parameters and the terms in the curly bracket are treated as observables. This is applied to study the end point in the T-μ phase diagram. In this case, the Monte Carlo simulation

is carried out where the parameters in α_0 include $\mu = 0$ and β_c which corresponds to the phase transition at temperature T_c. The parameter set α in the reweighted measure include $mu \neq 0$ and an adjusted β in the gauge action. The new β is determined from the Lee-Yang zeros so that one is following the transition line in the T-μ plane and the large change in the determinant ratio in the reweighting is compensated by the change in the gauge action to ensure reasonable overlap. This is shown to work to locate the transition line from $\mu = 0$ and $T = T_c$ down to the critical point on the 4^4 and $6^3 \times 4$ lattices with staggered fermions.[8]

While the multi-parameter reweighting is successful near the transition line, it is not clear how to extend it beyond this region, particularly the $T = 0$ case where one wants to keep the β and quark mass fixed while changing the μ. One still expects to face the overlap problem in the latter case. For large volumes, calculating the determinant ratio will be subjected to the same practical difficulty as discussed in the previous section 2.2.

3. Finite Baryon Density – A Canonical Ensemble Approach

We would like to propose an algorithm to overcome the overlap problem at zero temperature which is based on the canonical ensemble approach. To avoid the overlap problem, one needs to lock in a definite nonzero baryon sector so that the exponentially large contamination from the zero-baryon sector is excluded. To see this, we first note that the fermion determinant is a superposition of multiple quark loops of all sizes and shapes. This can be easily seen from the property of the determinant

$$\det M = e^{\mathrm{Tr}\log M} = 1 + \sum_{n=1} \frac{(\mathrm{Tr}\log M)^n}{n!}. \tag{8}$$

Upon a hopping expansion of $\log M$, $\mathrm{Tr}\log M$ represents a sum of single loops with all sizes and shapes. The determinant is then the sum of all multiple loops. The fermion loops can be separated into two classes. One is those which do not go across the time boundary and represent virtual quark-antiquark pairs; the other includes those which wraps around the time boundary which represent external quarks and antiquarks. The configuration with a baryon number one which entails three quark loops wrapping around the time boundary will have an energy M_B higher than that with zero baryon number. Thus, it is weighted with the probability $e^{-M_B N_t a_t}$ compared with the one with no net baryons. We see from the above discussion that the fermion determinant contains a superposition of sectors of all baryon numbers, positive, negative and zero. At zero temperature where $M_B N_t a_t \gg 1$, the zero baryon sector dominates and all the other baryon sectors are exponentially suppressed. It is obvious that to avoid the overlap problem, one needs to select a definite nonzero baryon number sector and stay in it throughout the Markov chain of updating configurations. To select a particular baryon sector from the determinant can be achieved by the following procedure:[9] first, assign an $U(1)$ phase factor $e^{-i\phi}$ to the links between the time slices t and $t+1$ so that the link $U/U^\dagger$ is multiplied

by $e^{-i\phi}/e^{i\phi}$; then the particle number projection can be carried out through the Fourier transformation of the fermion determinant like in the BCS theory

$$P_N = \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{-i\phi N} \det M[\phi] \tag{9}$$

where N is the net particle number, i.e. particle minus antiparticle. Note that all the virtual quark loops which do not reach the time boundary will have a net phase factor of unity; only those with a net N quark loops across the time boundary will have a phase factor $e^{i\phi N}$ which can contribute to the integral in Eq. (9). Since QCD in canonical formulation does not break $Z(3)$ symmetry, it is essential to take care that the ensemble is canonical with respect to triality. To this end, we shall consider the triality projection[9,10] to the zero triality sector

$$\det_0 M = \frac{1}{3}\sum_{k=0;\pm 1} \det M[\phi + k2\pi/3]. \tag{10}$$

This amounts to limiting the quark number N to a multiple of 3. Thus the triality zero sector corresponds to baryon sectors with integral baryon numbers.

Another essential ingredient to circumvent the overlap problem is to stay in the chosen nonzero baryon sector so as to avoid mixing with the zero baryon sector with exponentially large weight. This can be achieved by preforming the baryon number projection as described above *before* the accept/reject step in the Monte Carlo updating of the gauge configuration. If this is not done, the accepted gauge configuration will be biased toward the zero baryon sector and it is very difficult to project out the nonzero baryon sector afterwards. This is analogous to the situation in the nuclear many-body theory where it is known[13] that the variation after projection (Zeh-Rouhaninejad-Yoccoz method[14,15]) is superior than the variation before projection (Peierls-Yoccoz method[16]). The former gives the correct nuclear mass in the case of translation and yields much improved wave functions in mildly deformed nuclei than the latter.

To illustrate the overlap problem, we plot in Fig.1 $\mathrm{Tr}\log M[\phi]$ for a configuration of the $8^3 \times 12$ lattice with the Wilson action with $\beta = 6.0$ and $\kappa = 0.150$ which is obtained with 500 Z_2 noises. We see that the it is rather flat in ϕ indicating that the Fourier transform in Eq. (9) will mainly favor the zero baryon sector. On the other hand, at finite temperature, it is relatively easier for the quarks to be excited so that the zero baryon sector does not necessarily dominate other baryon sectors. Another way of seeing this is that the relative weighting factor $e^{-M_B N_t a_t}$ can be $O(1)$ at finite temperature. Thus, it should be easier to project out the nonzero baryon sector from the determinant. We plot in Fig. 2 a similarly obtained $\mathrm{Tr}\log M[\phi]$ for a configuration of the $8 \times 20^2 \times 4$ lattice with $\beta = 4.9$ and $\kappa = 0.182$. We see from the figure that there is quite a bit of wiggling in this case as compared to that in Fig. 1 indicating that it is easier to project out a nonzero baryon sector through the Fourier transform at finite temperature.

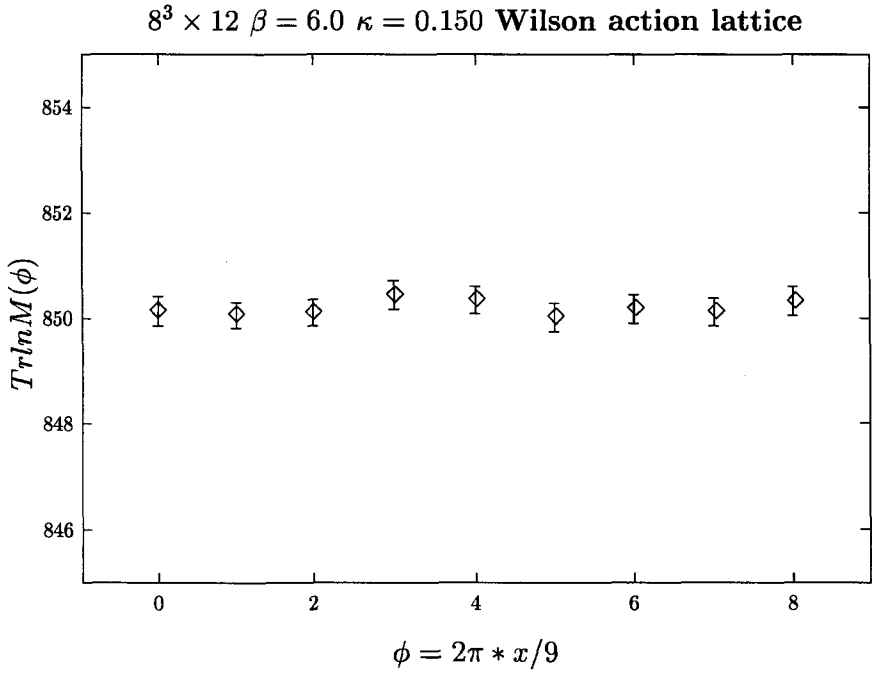

Fig. 1. $\mathrm{Tr}\log M[\phi]$ for a $8^3 \times 12$ configuration with Wilson action as a function of ϕ.

We should mention that while we think we can overcome the overlap problem and the determinant $\det M[\phi]$ is real in this approach, nevertheless in view of the fact that the Fourier transform in Eq. (9) involves the quark number N the canonical approach may still have the sign problem at the thermodynamic limit when N and V are very large. However, we think it might work for small N such as 3 or 6 for one or two baryons in a finite V. This should be a reasonable start for practical purposes.

While it is clear what the algorithm in the canonical approach entails, there are additional practical requirements for the algorithm to work. These include an unbiased estimation of the huge determinant in lattice QCD and, moreover, a Monte Carlo algorithm which accommodates the unbiased estimate of the probability. We shall discuss them in the following sections.

4. A Noisy Monte Carlo Algorithm

There are problems in physics which involve extensive quantities such as the fermion determinant which require V^3 steps to compute exactly. Problems of this kind with large volumes are not numerically applicable with the usual Monte Carlo algorithm which require an exact evaluation of the probability ratios in the accept/reject step.

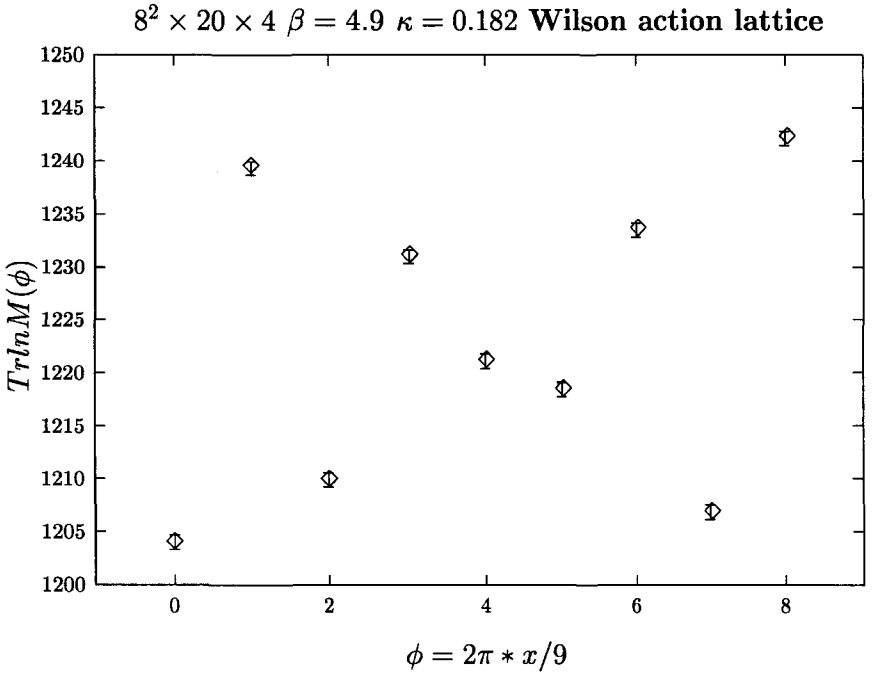

Fig. 2. Tr log $M[\phi]$ for a $8 \times 20^2 \times 4$ finite temperature configuration with dynamical fermion.

To address this problem, Kennedy and Kuti[11] proposed a Monte Carlo algorithm which admits stochastically estimated transition probabilities as long as they are unbiased. But there is a drawback. The probability could lie outside the interval between 0 and 1 since it is estimated stochastically. This probability bound violation will destroy detailed balance and lead to systematic bias. To control the probability violation with a large noise ensemble can be costly.

We propose a noisy Monte Carlo algorithm which avoids this difficulty with two Metropolis accept/reject steps. Let us consider a model with Hamiltonian $H(U)$ where U collectively denotes the dynamical variables of the system. The major ingredient of the new approach is to transform the noise for the stochastic estimator into stochastic variables. The partition function of the model can be written as

$$\begin{aligned} Z &= \int [DU]\, e^{-H(U)} \\ &= \int [DU][D\xi] P_\xi(\xi)\, f(U,\xi). \end{aligned} \tag{11}$$

where $f(U,\xi)$ is an unbiased estimator of $e^{-H(U)}$ from the stochastic variable ξ and P_ξ is the probability distribution for ξ.

The next step is to address the lower probability-bound violation. One first

notes that one can write the expectation value of the observable O as

$$\langle O\rangle = \int [DU][D\xi]\, P_\xi(\xi)\, O(U)\, \mathrm{sign}(f)\, |f(U,\xi)|/Z, \tag{12}$$

where $sign(f)$ is the sign of the function f. After redefining the partition function to be

$$Z = \int [DU][D\xi] P_\xi(\xi)\, |f(U,\xi)|, \tag{13}$$

which is semi-positive definite, the expectation of O in Eq. (12) can be rewritten as

$$\langle O\rangle = \langle O(U)\, \mathrm{sign}(f)\rangle / \langle \mathrm{sign}(f)\rangle. \tag{14}$$

As we see, the sign of $f(U,\xi)$ is not a part of the probability any more but a part in the observable. Notice that this reinterpretation is possible because the sign of $f(U,\xi)$ is a state function which depends on the configuration of U and ξ.

It is clear then, to avoid the problem of lower probability-bound violation, the accept/reject criterion has to be factorizable into a ratio of the new and old probabilities so that the sign of the estimated $f(U,\xi)$ can be absorbed into the observable. This leads us to the Metropolis accept/reject criterion which incidentally cures the problem of upper probability-bound violation at the same time. It turns out two accept/reject steps are needed in general. The first one is to propose updating of U via some procedure while keeping the stochastic variables ξ fixed. The acceptance probability P_a is

$$P_a(U_1,\xi \to U_2,\xi) \;=\; \min\Big(1, \frac{|f(U_2,\xi)|}{|f(U_1,\xi)|}\Big)\,. \tag{15}$$

The second accept/reject step involves the refreshing of the stochastic variables ξ according to the probability distribution $P_\xi(\xi)$ while keeping U fixed. The acceptance probability is

$$P_a(U,\xi_1 \to U,\xi_2) \;=\; \min\Big(1, \frac{|f(U,\xi_2)|}{|f(U,\xi_1)|}\Big)\,. \tag{16}$$

It is obvious that there is neither lower nor upper probability-bound violation in either of these two Metropolis accept/reject steps. Furthermore, it involves the ratios of separate state functions so that the sign of the stochastically estimated probability $f(U,\xi)$ can be absorbed into the observable as in Eq. (14).

Detailed balance can be proven to be satisfied and it is unbiased.[12] Therefore, this is an exact algorithm.

5. Noisy Monte Carlo with Fermion Determinant

One immediate application of NMC is lattice QCD with dynamical fermions. The action is composed of two parts – the pure gauge action $S_g(U)$ and a fermion action $S_F(U) = -\mathrm{Tr}\ln M(U)$. Both are functionals of the gauge link variables U.

To find out the explicit form of $f(U,\xi)$, we note that the fermion determinant can be calculated stochastically as a random walk process[17]

$$e^{\mathrm{Tr}\ln M} = 1 + \mathrm{Tr}\ln M(1 + \frac{\mathrm{Tr}\ln M}{2}(1 + \frac{\mathrm{Tr}\ln M}{3}(...))) \ . \tag{17}$$

This can be expressed in the following integral

$$e^{\mathrm{Tr}\ln M} = \int \prod_{i=1}^{\infty} d\eta_i \, P_\eta(\eta_i) \int_0^1 \prod_{n=2}^{\infty} d\rho_n$$
$$[1 + \eta_1^\dagger \ln M \eta_1 (1 + \theta(\rho_2 - \frac{1}{2})\eta_2^\dagger \ln M \eta_2 (1 + \theta(\rho_3 - \frac{2}{3})\eta_3^\dagger \ln M \eta_3 (...], \tag{18}$$

where $P_\eta(\eta_i)$ is the probability distribution for the stochastic variable η_i. It can be the Gaussian noise or the Z_2 noise ($P_\eta(\eta_i) = \delta(|\eta_i| - 1)$ in this case). The latter is preferred since it has the minimum variance.[18] ρ_n is a stochastic variable with uniform distribution between 0 and 1. This sequence terminates stochastically in finite time and only the seeds from the pseudo-random number generator need to be stored in practice. The function $f(U,\eta,\rho)$ (ξ in Eq. (11) is represented by two stochastic variables η and ρ here) is represented by the part of the integrand between the the square brackets in Eq. (18). One can then use the efficient Padé–Z_2 algorithm[19] to calculate the $\eta_i \ln M \eta_i$ in Eq. (18). We shall discuss this in the next section.

Finally, there is a practical concern that $\mathrm{Tr}\ln M$ can be large so that it takes a large statistics to have a reliable estimate of $e^{\mathrm{Tr}\ln M}$ from the series expansion in Eq. (18). In general, for the Taylor expansion $e^x = \sum x^n/n!$, the series will start to converge when $x^n/n! > x^{n+1}/(n+1)!$. This happens at $n = x$. For the case $x = 100$, this implies that one needs to have more than 100! stochastic configurations in the Monte Carlo integration in Eq. (18) in order to have a convergent estimate. Even then, the error bar will be very large. To avoid this difficulty, one can implement the following strategy. First, one notes that since the Metropolis accept/reject involves the ratio of exponentials, one can subtract a universal number x_0 from the exponent x in the Taylor expansion without affecting the ratio. Second, one can use a specific form of the exponential to diminish the value of the exponent. In other words, one can replace e^x with $(e^{(x-x_0)/N})^N$ to satisfy $|x - x_0|/N < 1$. The best choice for x_0 is $\overline{x}$, the mean of x. In this case, the variance of e^x becomes $e^{\delta^2/N} - 1$.

6. The Padé–Z_2 Method of Estimating Determinants

Now we shall discuss a very efficient way of estimating the fermion determinant stochastically.[19]

6.1. *Padé approximation*

The starting point for the method is the Padé approximation of the logarithm function. The Padé approximant to $\log(z)$ of order $[K,K]$ at z_0 is a rational function

$N(z)/D(z)$ where deg $N(z)$ = deg $D(z) = K$, whose value and first $2K$ derivatives agree with $\log z$ at the specified point z_0. When the Padé approximant $N(z)/D(z)$ is expressed in partial fractions, we obtain

$$\log z \approx b_0 + \sum_{k=1}^{K} \left(\frac{b_k}{z + c_k} \right), \tag{19}$$

whence it follows

$$\log \det \mathbf{M} = \mathrm{Tr}\ \mathrm{logM} \approx b_0 \mathrm{TrI} + \sum_{k=1}^{K} b_k \cdot \mathrm{Tr}(\mathbf{M} + c_k \mathbf{I})^{-1}. \tag{20}$$

The Padé approximation is not limited to the real axis. As long as the function is in the analytic domain, i. e. away from the cut of the log, say along the negative real axis, the Padé approximation can be made arbitrarily accurate by going to a higher order $[K, K]$ and a judicious expansion point to cover the eigenvalue domain of the problem.

6.2. *Complex Z_2 noise trace estimation*

Exact computation of the trace inverse for $N \times N$ matrices is very time consuming for matrices of size $N \sim 10^6$. However, the complex Z_2 noise method has been shown to provide an efficient stochastic estimation of the trace.[18,20,21] In fact, it has been proved to be an optimal choice for the noise, producing a *minimum* variance.[22]

The complex Z_2 noise estimator can be briefly described as follows.[18,22] We construct L noise vectors $\eta^1, \eta^2, \cdots, \eta^L$ where $\eta^j = \{\eta_1^j, \eta_2^j, \eta_3^j, \cdots, \eta_N^j\}^T$, as follows. Each element η_n^j takes one of the four values $\{\pm 1, \pm i\}$ chosen independently with equal probability. It follows from the statistics of η_n^j that

$$E[< \eta_n >] \equiv E[\frac{1}{L} \sum_{j=1}^{L} \eta_n^j] = 0, \qquad E[< \eta_m^\star \eta_n >] \equiv E[\frac{1}{L} \sum_{j=1}^{L} \eta_m^{\star j} \eta_n^j] = \delta_{mn}. \tag{21}$$

The vectors can be used to construct an unbiased estimator for the trace inverse of a given matrix M as follows:

$$\begin{aligned} E[< \eta^\dagger \mathbf{M}^{-1} \eta >] &\equiv E[\frac{1}{L} \sum_{j=1}^{L} \sum_{m,n=1}^{N} \eta_m^{\star j} M_{m,n}^{-1} \eta_n^j] \\ &= \sum_{n}^{N} M_{n,n}^{-1} + (\sum_{m \neq n}^{N} M_{m,n}^{-1})[\frac{1}{L} \sum_{j}^{L} \eta_m^{\star j} \eta_n^j] \\ &= \mathrm{Tr}\ \mathbf{M}^{-1}. \end{aligned}$$

The variance of the estimator is shown to be[22]

$$\begin{aligned} \sigma_M^2 &\equiv \mathrm{Var}[< \eta^\dagger \mathbf{M}^{-1} \eta >] = \mathrm{E}\left[| < \eta^\dagger \mathbf{M}^{-1} \eta > - \mathrm{Tr}\ \mathbf{M}^{-1} |^2 \right] \\ &= \frac{1}{L} \sum_{m \neq n}^{N} M_{m,n}^{-1} (M_{m,n}^{-1})^\star = \frac{1}{L} \sum_{m \neq n}^{N} |M_{m,n}^{-1}|^2 . \end{aligned}$$

The stochastic error of the complex Z_2 noise estimate results only from the off-diagonal entries of the inverse matrix (the same is true for Z_n noise for any n). However, other noises (such as Gaussian) have additional errors arising from diagonal entries. This is why the Z_2 noise has minimum variance. For example, it has been demonstrated on a $16^3 \times 24$ lattice with $\beta = 6.0$ and $\kappa = 0.148$ for the Wilson action that the Z_2 noise standard deviation is smaller than that of the Gaussian noise by a factor of 1.54.[18]

Applying the complex Z_2 estimator to the expression for the Trlog**M** in Eq. (20), we find

$$\begin{aligned}
&\sum_k b_k \mathrm{Tr}(M + c_k)^{-1} \\
&\approx \frac{1}{L} \sum_k^K \sum_j^L b_k \eta^{j\dagger} (M + c_k)^{-1} \eta^j \\
&= \frac{1}{L} \sum_j^L \sum_{k=1}^K b_k \eta^{j\dagger} \xi^{k,j},
\end{aligned} \tag{22}$$

where $\xi^{k,j} = (M + c_k \mathbf{I})^{-1} \eta^j$ are the solutions of

$$(M + c_k \mathbf{I}) \xi^{k,j} = \eta^j, \tag{23}$$

Since $M + c_k \mathbf{I}$ are shifted matrices with constant diagonal matrix elements, Eq. (23) can be solved collectively for all values of c_k within one iterative process by several algorithms, including the Quasi-Minimum Residual (QMR),[23] Multiple-Mass Minimum Residual (M^3 R),[24] and GMRES.[25] We have adopted the M^3 R algorithm, which has been shown to be about 2 times faster than the conjugate gradient algorithm, and the overhead for the multiple c_k is only 8%.[26] The only price to pay is memory: for each c_k, a vector of the solution needs to be stored. Furthermore, one observes that $c_k > 0$. This improves the conditioning of $(\mathbf{M} + c_k \mathbf{I})$ since the eigenvalues of **M** have positive real parts. Hence, we expect faster convergence for column inversions for Eq. (23).

In the next section, we describe a method which significantly reduces the stochastic error.

6.3. *Improved PZ estimation with unbiased subtraction*

In order to reduce the variance of the estimate, we introduce a suitably chosen set of traceless $N \times N$ matrices $\mathbf{Q}^{(p)}$, i.e. which satisfy $\sum_{n=1}^N \mathbf{Q}^{(p)}_{n,n} = 0, p = 1 \cdots P$. The expected value and variance for the modified estimator $< \eta^\dagger (\mathbf{M}^{-1} - \sum_{p=1}^P \lambda_p \mathbf{Q}^{(p)}) \eta >$ are given by

$$E[< \eta^\dagger (\mathbf{M}^{-1} - \sum_{p=1}^P \lambda_p \mathbf{Q}^{(p)})) \eta >] = \mathrm{Tr}\ \mathbf{M}^{-1}\ , \tag{24}$$

$$\Delta_M(\lambda) = \mathrm{Var}[< \eta^\dagger(\mathbf{M}^{-1} - \sum_{\mathrm{p}=1}^{\mathrm{P}} \lambda_{\mathrm{p}} \mathbf{Q}^{(\mathrm{p})})\eta >]$$
$$= \frac{1}{L} \sum_{m \neq n} |\mathbf{M}^{-1}_{m,n} - \sum_{p=1}^{P} \lambda_p \mathbf{Q}^{(p)}_{m,n})|^2 \,, \tag{25}$$

for any values of the real parameters λ_p. In other words, introducing the matrices $\mathbf{Q}^{(p)}$ into the estimator produces no bias, but may reduce the error bars if the $\mathbf{Q}^{(p)}$ are chosen judiciously. Further, λ_p may be varied at will to achieve a minimum variance estimate: this corresponds to a least-squares fit to the function $\eta^\dagger \mathbf{M}^{-1} \eta$ sampled at points η_j, $j = 1 \cdots L$, using the fitting functions $\left\{1, \eta^\dagger \mathbf{Q}^{(p)} \eta\right\}$, $p = 1 \cdots P$.

We now turn to the question of choosing suitable traceless matrices $\mathbf{Q}^{(p)}$ to use in the modified estimator. One possibility for the Wilson fermion matrix $\mathbf{M} = \mathbf{I} - \kappa \mathbf{D}$ is suggested by the hopping parameter — κ expansion of the inverse matrix,

$$(\mathbf{M} + c_k \mathbf{I})^{-1} = \frac{1}{\mathbf{M} + c_k \mathbf{I}} = \frac{1}{(1 + c_k)(\mathbf{I} - \frac{\kappa}{(1+c_k)} \mathbf{D})}$$
$$= \frac{\mathbf{I}}{1 + c_k} + \frac{\kappa}{(1 + c_k)^2} \mathbf{D} + \frac{\kappa^2}{(1 + c_k)^3} \mathbf{D}^2 + \frac{\kappa^3}{(1 + c_k)^4} \mathbf{D}^3 + \cdots \,. \tag{26}$$

This suggests choosing the matrices $\mathbf{Q}^{(p)}$ from among those matrices in the hopping parameter expansion which are traceless:

$$\mathbf{Q}^{(1)} = \frac{\kappa}{(1 + c_k)^2} \mathbf{D},$$
$$\mathbf{Q}^{(2)} = \frac{\kappa^2}{(1 + c_k)^3} \mathbf{D}^2,$$
$$\mathbf{Q}^{(3)} = \frac{\kappa^3}{(1 + c_k)^4} \mathbf{D}^3,$$
$$\mathbf{Q}^{(4)} = \frac{\kappa^4}{(1 + c_k)^5} (\mathbf{D}^4 - \mathrm{Tr}\mathbf{D}^4),$$
$$\mathbf{Q}^{(5)} = \frac{\kappa^5}{(1 + c_k)^6} \mathbf{D}^5,$$
$$\mathbf{Q}^{(6)} = \frac{\kappa^6}{(1 + c_k)^7} (\mathbf{D}^6 - \mathrm{Tr}\mathbf{D}^6),$$
$$\mathbf{Q}^{(2r+1)} = \frac{\kappa^{2r+1}}{(1 + c_k)^{2r+2}} \mathbf{D}^{2r+1}, \qquad r = 3, 4, 5, \cdots \,.$$

It may be verified that all of these matrices are traceless. In principle, one can include all the even powers which entails the explicit calculation of all the allowed loops in $\mathrm{Tr}D^{2r}$. In this manuscript we have only included $\mathbf{Q}^{(4)}$, $\mathbf{Q}^{(6)}$, and $\mathbf{Q}^{(2r+1)}$.

6.4. *Computation of Tr log M*

Our numerical computations were carried out with the Wilson action on the $8^3 \times 12$ ($N = 73728$) lattice with $\beta = 5.6$. We use the HMC with pseudofermions to generate gauge configurations. With a cold start, we obtain the fermion matrix $\mathbf{M}_1$ after the plaquette becomes stable. The trajectories are traced with $\tau = 0.01$ and 30 molecular dynamics steps using $\kappa = 0.150$. $\mathbf{M}_2$ is then obtained from $\mathbf{M}_1$ by an accepted trajectory run. Hence $\mathbf{M}_1$ and $\mathbf{M}_2$ differ by a continuum perturbation, and $\log[\det \mathbf{M}_1 / \det \mathbf{M}_2] \sim \mathcal{O}(1)$.

We first calculate $\log \det \mathbf{M}_1$ with different orders of Padé expansion around $z_0 = 0.1$ and $z_0 = 1.0$. We see from Table 1 that the 5th order Padé does not give the same answer for two different expansion points, suggesting that its accuracy is not sufficient for the range of eigenvalues of $\mathbf{M}_1$. Whereas, the 11th order Padé gives the same answer within errors. Thus, we shall choose P[11,11](z) with $z_0 = 0.1$ to perform the calculations from this point on.

Table 1. Unimproved and improved PZ estimates for log [det$\mathbf{M}_1$] with 100 complex Z_2 noise vectors. $\kappa = 0.150$.

$P[K,K](z)$	$K =$	5	7	9	11
$z_0 = 0.1$	Original:	473(10)	774(10)	796(10)	798(10)
	Improved:	487.25(62)	788.17(62)	810.83(62)	812.33(62)
$z_0 = 1.0$	Original:	798(10)	798(10)	798(10)	799(10)
	Improved:	812.60(62)	812.37(62)	812.36(62)	812.37(62)

In Table 2, we give the results of improved estimations for $\mathrm{Tr} \log \mathbf{M}_1$. We see that the variational technique described above can reduce the data fluctuations by more than an order of magnitude. For example, the unimproved error $\delta_0 = 5.54$ in Table 2 for 400 Z_2 noises is reduced to $\delta_{11} = 0.15$ for the subtraction which includes up to the $\mathbf{Q}^{11}$ matrix. This is 37 times smaller. Comparing the central values in the last row (i.e. the 11^{th} order improved) with that of unimproved estimate with 10,000 Z_2 noises, we see that they are the same within errors. This verifies that the variational subtraction scheme that we employed does not introduce biased errors. The improved estimates of $\mathrm{Tr} \log \mathbf{M}_1$ from 50 Z_2 noises with errors δ_r from Table 2 are plotted in comparison with the central value of the unimproved estimate from 10,000 noises in Fig. 3.

7. Implementation of the Kentucky Noisy Monte Carlo Algorithm

We have recently implemented the above Noisy Monte Carlo algorithm to the simulation of lattice QCD with dynamical fermions by incorporating the full determinant directly.[28] Our algorithm uses pure gauge field updating with a shifted gauge

Table 2. Central values for improved stochastic estimation of log[det M_1] and rth–order improved Jackknife errors δ_r are given for different numbers of Z_2 noise vectors. κ is 0.150 in this case.

# Z_2	50	100	200	400	600	800	1000	3000	10000
0^{th}	802.98	797.98	810.97	816.78	815.89	813.10	816.53	813.15	812.81
δ_0	±14.0	±9.81	±7.95	±5.54	±4.47	±3.83	±3.41	±1.97	±1.08
1^{st}	807.89	811.21	814.13	815.11	814.01	814.62	814.97	—	—
δ_1	±4.65	±3.28	±2.48	±1.84	±1.50	±1.29	±1.12	-	-
2^{nd}	813.03	812.50	811.99	812.86	811.87	812.89	813.04	—	—
δ_2	±2.46	±1.65	±1.34	±1.01	±0.83	±0.72	±0.64	-	-
3^{rd}	812.07	812.01	811.79	812.44	812.18	812.99	813.03	—	—
δ_3	±1.88	±1.31	±1.01	±0.74	±0.58	±0.51	±0.44	-	-
4^{th}	812.28	812.52	812.57	812.86	812.85	813.25	813.40	—	—
δ_4	±1.20	±0.94	±0.68	±0.48	±0.39	±0.35	±0.30	-	-
5^{th}	813.50	813.07	813.36	813.70	813.47	813.54	813.50	—	—
δ_5	±0.82	±0.62	±0.47	±0.34	±0.29	±0.25	±0.22	-	-
6^{ts}	813.54	813.23	813.22	813.28	813.20	813.37	813.26	—	—
δ_6	±0.67	±0.49	±0.35	±0.25	±0.21	±0.18	±0.16	-	-
7^{ts}	814.18	813.74	813.44	813.42	813.39	—	—	—	—
δ_7	±0.44	±0.36	±0.26	±0.19	±0.16	-	-	-	-
9^{th}	813.77	813.62	813.49	813.40	813.43	—	—	—	—
δ_9	±0.40	±0.30	±0.22	±0.16	±0.14	-	-	-	-
11^{th}	813.54	813.41	813.45	813.44	813.43	—	—	—	—
δ_{11}	±0.38	±0.27	±0.21	±0.15	±0.13	-	-	-	-

coupling to minimize fluctuations in the trace log is the Wilson Dirac matrix. It gives the correct results as compared to the standard Hybrid Monte Carlo simulation. However, the present simulation has a low acceptance rate due to the pure gauge update and results in long autocorrelations. We are in the process of working out an alternative updating scheme with molecular dynamics trajectory to include the feedback of the determinantal effects on the gauge field which should be more efficient than the pure gauge update.

8. Summary

After reviewing the finite density algorithm for QCD with the chemical potential, we propose a canonical approach by projecting out the definite baryon number sector from the fermion determinant and stay in the sector throughout the Monte Carlo updating. This should circumvent the overlap problem. In order to make the algorithm practical, one needs an efficient way to estimate the huge fermion determinant and a Monte Carlo algorithm which admits unbiased estimates of the probability without upper unitarity bound violations. These are achieved with the Padé–Z_2 estimate of the determinant and the Noisy Monte Carlo algorithm. So far, we have implemented the Kentucky Noisy Monte Carlo algorithm to incorporate dynamical fermions in QCD on a relatively small lattice and medium heavy quark

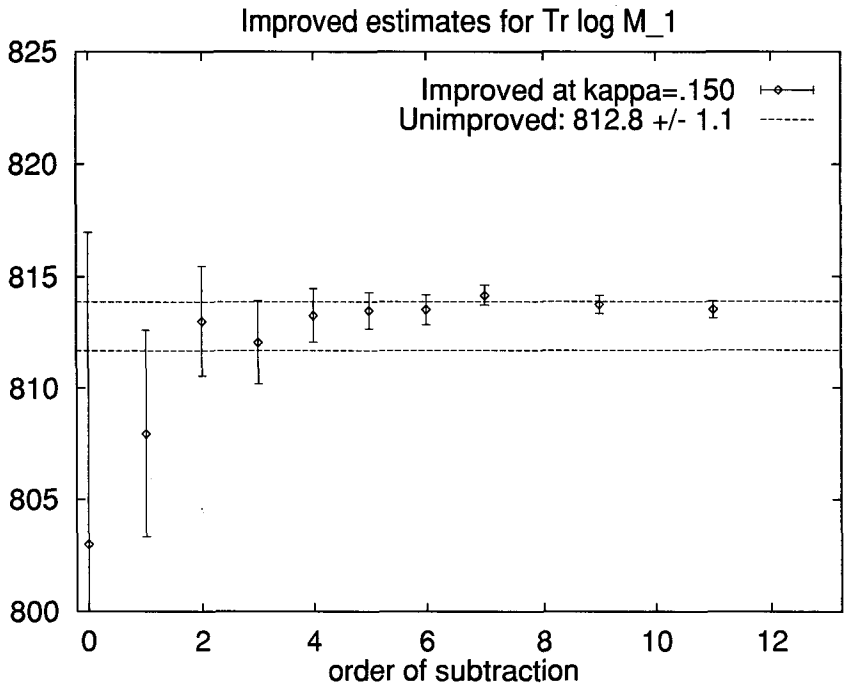

Fig. 3. The improved PZ estimate of $\mathrm{Tr}\log \mathbf{M}_1$ with 50 noises as a function of the order of subtraction and compared to that of unimproved estimate with 10,000 noises. The dashed lines are drawn with a distance of 1 σ away from the central value of the unimproved estimate.

based on pure gauge updating. As a next step, we will work on a more efficient updating algorithm and project out the baryon sector to see if the finite density algorithm proposed here will live up to its promise.

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SHORT-TIME BEHAVIORS OF LONG-RANGED INTERACTIONS*

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Recent results on the short-time behaviors of a few models possessing a common feature of long-ranged interaction will be summarized. For the disorder initial state, the initial order increase is observed for each model in a heat-bath at the critical temperature. The dynamic exponents are calculated. For arbitrary initial order and environment temperature, universal characteristic functions are introduced in order to generalize the scaling relations. Remarkable consistence between the theoretic renormalization group results and the simulations are found in the long-range regime.

The short-time phenomena are those which happen at the times just after a microscopic time-scale $t_{\rm mic}$ needed for a system to forget its microscopic details, and much smaller than the macroscopic time scale $t_{\rm mac} \sim \tau^{-\nu z}$. In this time regime, the system still remembers the macroscopic feature of the initial state. Since the pioneer work of H.K. Janssen et al.,[1] universal short-time scaling has been found in a variety of different models (for a review, see Ref. 2). For initial states of zero correlation length and zero (or very small) initial order, the order increases in the short-time regime with a power law $t^{\theta'}$ where θ' is a characteristic exponent of the short-time regime.

Recently, we generalized the results of Ref. 1 to systems with long-ranged interactions,[3] with an anisotropic cubic term,[4,5] and with impurities.[6,7] Hopefully these models could describe some realistic systems. On the other hand, we expect that the analytical calculations for long-ranged interactions based on expansions around the upper critical dimensions are more reliable in physical dimensions since

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the longer range of interaction the lower critical dimension($\epsilon = 2\sigma - d$ with σ the decay power of the interaction).

A common observation is that the critical exponent θ' depends on the range of interactions. In the theoretic renormalization group calculation, the crossover from the long-ranged interaction(LRI) to the short-ranged interaction(SRI) is subtle and needs extra effort[8] since the fixed point of LRI does not continuously approach that of the SRI as the interaction range decreases. There is a competition between two fixed points in the regime of the weakly long-ranged interaction. The exactly solvable kinetic spherical model provides a concrete example for the short-time behavior of LRI.[9]

In one dimension, Monte Carlo simulation is possible. The theoretic results of LRI can be checked numerically. In order to have a picture for the crossover of LRI and SRI, we simulate an adsorption-desorption model which has a dynamic phase transition even for SRI.

1. Kinetic spherical model

The Hamiltonian of the spherical model is

$$H = \frac{\alpha}{2}\sum_i S_i^2 - \frac{\beta}{2}\sum_{ij} J_{ij}S_iS_j \tag{1}$$

with the constraint

$$\sum_i S_i^2 = N \tag{2}$$

where i, j are labels of lattice sites, N is the total number of spins. In the dynamic process, α is a time-dependent Lagrange multiplier corresponding to the constraint. Joyce[10] first studied the static spherical model with long-ranged ferromagnetic interactions. In a d-dimensional lattice,

$$J_{ij} = J_0 r_{ij}^{-(d+s)} / \sum_j r_{ij}^{-(d+s)}$$

with $0 < s < 2$ for long-ranged interactions and $s = \sigma$, while $s > 2$ for short-ranged interactions and $\sigma = 2$. Where r_{ij} is the distance between the sites i and j.

The Langevin equation for this constrained spin system is

$$\frac{\partial S_i}{\partial t} = -\lambda\alpha S_i + \lambda\beta\sum_j J_{ij}S_j + \eta_i \tag{3}$$

where λ is the kinetic coefficient and η_i being Gaussian white noises.

A remarkable observation is that the ordering process (governed by the zero-temperature fixed point) and the critical dynamics (governed by the critical fixed point) can be uniquely described by a characteristic function. The universal characteristic function for arbitrary initial order[11] is also found analytically. The generalized scaling relation for the relative order $m_r = (m(t) - m(\infty))/m(\infty)$ are

$$m_r(t, T', m_0) = m_r(b^{-z}t, \epsilon(b, T'), \varphi(b, m_0)) \tag{4}$$

with $T' = T/T_c$. The characteristic function $\varphi(b, m_0)$ is given by

$$\varphi(b, m_0) = [b^{-z}(m_0^{-2} - 1) + 1]^{-\frac{1}{2}} \tag{5}$$

The function $\epsilon(b, T')$ is the solution of the following equation

$$\frac{\epsilon^k(b, T')}{1 - \epsilon(b, T')} = b^{-z(k-1)} \frac{T'^k}{1 - T'} \tag{6}$$

Where $k = d/\sigma$ and $z = \sigma$. At the critical point $\epsilon(b, 1) = 1$. As $m_0 \to 0$, $\varphi(b, m_0) \to b^{z/2} m_0$, one attains

$$m(t) = m_0 t^{1 - \frac{k}{2}} \tag{7}$$

That is $\theta' = 1 - k/2$.

By use of the characteristic functions, the response propagator and correlation function have the generalized scaling forms which are valid for the times larger than t_{mic} and for an arbitrary m_0,

$$G_p(t, t', m_0) = p^{-2+\eta+z} \widetilde{h}(p\xi(t), p\xi(t'), \epsilon(p^{-1}, T'), \varphi(p^{-1}, m_0)) \tag{8}$$

$$C_p(t, t', m_0) = p^{-2+\eta} \widetilde{g}(p\xi(t), p\xi(t'), \epsilon(p^{-1}, T'), \varphi(p^{-1}, m_0)) \tag{9}$$

where the domain size[12] $\xi(t) \sim t^{\rho}$ with $\rho = 1/\sigma$, $\widetilde{h}$ and $\widetilde{g}$ are two universal functions.

2. Dynamic Ginzburg–Landau model

The Ginzburg–Landau model with anisotropic cubic term has a hamiltonian

$$H[s] \equiv \int d^d x \left\{ \frac{a}{2} (\bigtriangledown s)^2 + \frac{b}{2} (\bigtriangledown^{\frac{\sigma}{2}} s)^2 + \frac{\tau}{2} s^2 + \frac{g_i}{4!} (s^2)^2 + \frac{g_a}{4!} \sum_{\alpha=1}^{n} (s^\alpha)^4 \right\} \tag{10}$$

where $s = (s^\alpha)$ are n-component order parameter fields; g_i and g_a are the coupling constants for the isotropy and the anisotropy respectively. The SRI model corresponds to $a = 1$ and $b = 0$, whereas for the pure LRI model $\sigma < 2$, $a = 0$ and $b = 1$. The long-time relaxation behavior of the model has been extensively studied.[10,13–15] Here we will concentrate on the short-time behavior.

The dynamics is given by the Langevin equation

$$\partial_t s^\alpha(x, t) = -\lambda \frac{\delta H[s]}{\delta s^\alpha(x, t)} + \xi^\alpha(x, t) \tag{11}$$

where λ is the kinetic coefficient. The random forces $\xi = (\xi^\alpha)$ are assumed to be Gaussian distributed.

The anisotropic cubic fixed point is stable when $n > n_c = 4 - 2D_\sigma \epsilon$ where $D_\sigma = \psi(1) - 2\psi(\sigma/2) + \psi(\sigma)$ with $\psi(x)$ being the logarithmic derivative of the gamma function. In the two-loop level, we attained for $n > n_c$

$$\theta' = \frac{\epsilon(n-1)}{3\sigma n} \left\{ 1 - \left[\frac{n^2 - 18n + 24}{6n^2} D_\sigma - \frac{2(n+2)}{3\sigma n} (\ln 2 - \sigma B_\sigma) \right] \epsilon \right\} . \tag{12}$$

Pure F.P.	SRQI F.P.	
$n > 4,\ \delta < \delta_1$	$1 < n < 4,\ \delta < \delta_2$	$n = 1,\ \delta < \delta_3$
$\frac{n+2}{2\sigma(n+8)}\epsilon$	$\frac{\epsilon}{4\sigma}$	0
LRQI F.P.		
$n \neq 1,\ \delta > \delta_1$ or δ_2	$n = 1,\ \delta > \delta_3$	$n = 1,\ \delta \sim \epsilon^{1/2}$
$\frac{2(n-1)\rho+3n\epsilon}{2\sigma(5n+4)}$	$\frac{\epsilon}{6\sigma}$	0

Here we have introduced

$$B_\sigma \equiv K_{2\sigma}^{-1} \int \frac{d^{2\sigma}x}{(2\pi)^{2\sigma}} [1 + x^\sigma + (\mathbf{e} + \mathbf{x})^\sigma]^{-2} x^{-\sigma}$$

with $\mathbf{e}$ a unit vector in the 2σ-dimensional space, $K_{2\sigma} = 2^{1-2\sigma}/[\pi^\sigma \Gamma(\sigma)]$.

For $n < n_c$ one has the same result as in the isotropic model ($g_a = 0$ and $\sigma < 2 - \eta_{sr}$, with η_{sr} the Fisher exponent at the SRI fixed point)

$$\begin{aligned} \theta' &= -\frac{\eta_s + \eta_{\bar{s}} + \eta_0}{2z} \\ &= \frac{\epsilon(n+2)}{2\sigma(n+8)} \left\{ 1 + \left[\frac{7n+20}{(n+8)^2} D_\sigma + \frac{12(\ln 2 - \sigma B_\sigma)}{\sigma(n+8)} \right] \epsilon \right\} \end{aligned} \tag{13}$$

A discussion on the weakly long-range regime of $2 - \eta_{sr} < \sigma < 2$ is given in Ref. 8.

3. Dynamic Ginzburg-Landau model with impurities

The Hamiltonian with both LRI and long-range quenched impurities(LRQI) is defined as

$$H[s] = \int d^d x \left\{ \frac{1}{2} (\nabla^{\frac{\sigma}{2}} s)^2 + \frac{\tau}{2} s^2 + \frac{g}{4!} (s^2)^2 + \frac{1}{2} \phi s^2 \right\} \tag{14}$$

The static random-impurity noises $\phi(x)$ describe the quenched disorders (random temperatures) and satisfy the following configurational averages

$$\langle \phi(x) \rangle_{av} = 0,\ \langle \phi(x)\phi(x') \rangle_{av} = [g_1 + g_2(-\nabla^2)^{-\rho/2}]\delta(x - x'),$$

where g_1 corresponds to the SRQI, whereas g_2 represents the LRQI.[16]

Table 1 gives the results for $d < 2\sigma$ to the first order in ϵ and ρ, or $\epsilon^{1/2}$. We have defined $\delta = \epsilon + \rho$, $\delta_1 = 2(n+2)\epsilon/(n+8)$, $\delta_2 = 3n\epsilon/[4(n-1)]$, and $\delta_3 = \frac{2}{3}(\epsilon/D_\sigma)^{1/2}$. For $n < 4$, the impurities are relevant because of the extended Harris criterion.[16] It is remarkable that the exponent θ' at the SRQI fixed point does not depend upon n for $1 < n < 4$ and vanishes for $n = 1$.

Rich scaling patterns are found at the upper critical dimension $d = 2\sigma$. They are (1) the impurity irrelevant phase of $n > 4$

$$m(t) = m_0 \left(\ln \frac{t}{t_0} \right)^{\frac{n+2}{2(n+8)}} \tag{15}$$

(2) the short-range disorder phase of $1 < n < 4$

$$m(t) = m_0 \left(\ln \frac{t}{t_1}\right)^{-\frac{(4-n)}{8(n-1)}} \left(\ln \frac{t}{t_0'}\right)^{\frac{n+2}{8(n-1)}} \tag{16}$$

(3) Ising short-range disorder of $n = 1$. In this case, the one-loop exponents are non-universal due to the degeneracy of β-function.[6] The two-loop result is

$$m(t) \sim m_0 \exp\left\{\left(\frac{2}{9\sigma D_\sigma}\right)^{1/2} \left[(\ln(t/t_2))^{1/2} - (\ln(t/t_0''))^{1/2}\right]\right\} \tag{17}$$

In the above three equations, t_0, t_0', t_0'' and t_1, t_2 are microscopic time-scales

4. Monte Carlo simulations

Monte Carlo simulation for 1-d LRI Ising model is being done.[17] The preliminary results are encouraging. For $\sigma = 0.7$, the simulation gives $\theta' = 0.1648$. This value agrees with our theoretical prediction $\theta' = 0.1673$. Contrast to this, the theoretic values of θ' of the SRI are 0.131 for $d = 3$ and 0.356 for $d = 2$, whereas the results of Monte Carlo are 0.104 and 0.191 respectively.

We also simulated a one-dimensional adsorption-desorption process (ADP) which is an irreversible non-equilibrium model. Each site could be either occupied(denoted by $s_i = 1$) or not-occupied(denoted by $s_i = 0$). The possibility of adsorption is λ (provided the site under consideration is vacant). The desorption has a long-ranged correlated probability

$$w(s_i = 1 \to s_i = 0) = A \sum_{|j-i|>0} (1 - s_j) \frac{1}{|j-i|^{1+\sigma}} \tag{18}$$

By choosing A, the desorption probability is normalized to unity for the state that all sites are vacant. The critical exponent θ' versus the interaction range parameter σ is plotted in Figure 1. Within the statistical error, one sees that the exponent of $\sigma \geq 2$ has the same value as that of the short-range model.[18] In the last reference, the characteristic function is also discussed.

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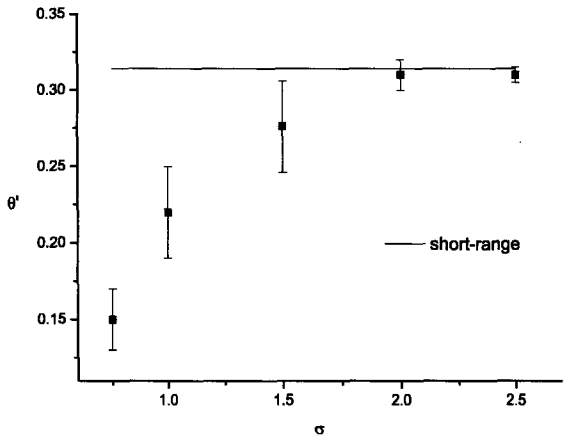

Fig. 1. Monte Carlo results of θ' of the one-dimensional absorption-desorption process for various values of σ. The straight line is the result of the short-ranged interaction model.

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GENERALIZED SUPERSYMMETRIES AND COMPOSITE STRUCTURE IN M-THEORY*

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We describe generalized $D = 11$ Poincaré and conformal supersymmetries. The corresponding generalization of twistor and supertwistor framework is outlined with $OSp(1|64)$ superspinors describing BPS preons. The $\frac{k}{32}$ BPS states as composed out of $n = 32 - k$ preons are introduced, and basic ideas concerning BPS preon dynamics is presented. The lecture is based on results obtained by J.A. de Azcarraga, I. Bandos, J.M. Izquierdo and the author.[1]

1. Introduction

M-theory has been proposed as a hypothetical quantum theory describing elementary level of matter, which should incorporate and possibly explain various properties of "new string theory" (for review see e.g. Refs. 2, 4). One of the features of such new theory of fundamental interactions should be the appearance of many extended elementary objects (p-(super)branes, D-(super)branes etc.) related with each other via duality/dimensional reductions net. Such a variety of basic objects in the theory makes sensible a search for some underlying composite structure.

The basic dynamical degrees of freedom in M-theory yet are not known—there were presented only some proposals usually related with $D = 11$ space-time geometry. We postulate that the composite structure of M-theory should be formulated in terms of new degrees of freedom related with new geometry. Because M-theory is supersymmetric, and supersymmetry reveals more elementary nature of spinorial objects, we shall postulate that the basic fundamental geometric structure in M-theory is spinorial.

The only well-known part of the description of M-theory is algebraic. Assuming that M-theory lives in $D = 11$ (this assumption is consistent with description of $D = 11$ SUGRA as the low energy limit of M-theory) we can postulate the following

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basic $D = 11$ M-superalgebra[a]

$$\{Q_r, Q_s\} = Z_{rs}(\Gamma_\mu C)_{rs} P^k + C\Gamma_{[\mu\nu]}C)_{rs} Z^{[\mu\nu]} + \left(\Gamma_{[\mu_1-\mu_s]}C\right)_{rs} Z^{[\mu_1\ldots\mu_5]} . \quad (1)$$

where $\mu, \nu = 0, 1, \ldots 10$, $r, s = 1, \ldots 32$. The collection of 528 Abelian generators Z_{rs} ($Z_{rs} = Z_{sr}$) describes the generalized momenta in M-theory. Introducing dual generalized coordinate space

$$X_{rs} = (\Gamma_\mu C)_{rs} X^\mu + \left(\Gamma_{[\mu\nu]}C\right)_{rs} X^{[\mu\nu]} + \left(\Gamma_{[\mu_5-\mu_s]}C\right)_{rs} X^{[\mu_1\ldots\mu_5]} , \quad (2)$$

we obtain large generalized phase space, with coordinates and positions described by the adjoint representations of $Sp(32)$ algebra.

Let us recall the assumption of Penrose twistor formalism in $D = 4$[8,11] that basic spinorial degrees of freedom in twistorial theory of elementary particles are described by N twistors ($i = 1 \ldots N$)

$$t_\alpha^{(i)} = \left(\lambda_A^{(i)}, \omega^{(i)\dot{A}}\right) , \quad (3)$$

where $\lambda_A^{(i)}$, $\omega^{(i)\dot{A}}$ ($A = 1, 2$) are the pairs of $D = 4$ Weyl spinors. The following formula for the composite four-momentum is assumed[9,11]

$$P_{A\dot{B}} = \sum_{i=1}^{N} \lambda_A^{(i)} \overline{\lambda}_{\dot{B}}^{(i)} , \quad (4)$$

where $P_{A\dot{B}} = \frac{1}{2}\sigma_{A\dot{B}}^\mu P_\mu$. We shall propose analogous formula in $D = 11$ for generalized momenta

$$Z_{rs} = \sum_{i=1}^{N} \lambda_r^{(i)} \lambda_s^{(i)} , \quad (5)$$

where λ_r ($r = 1 \ldots 32$) are $D = 11$ real Majorana spinors. In $D = 4$ the twistors (3) are the fundamental representations of the spinorial covering $SU(2, 2)$ of $D = 4$ conformal algebra ($SU(2, 2) = SO(\overline{4, 2})$). In $D = 11$ there exists only minimal conformal spinorial algebra[12,14] describing the classical real algebra $Sp(64)$, containing $D = 11$ conformal algebra

$$SO(\overline{11, 2}) \subset Sp(64; R) . \quad (6)$$

In Sect. 2 we shall consider the generalization of $D = 11$ Poincaré and conformal superalgebras, supersymmetrizing the minimal $D = 11$ conformal spinorial algebra. In Sect. 3 we shall introduce in $D = 11$ the generalization of twistor and supertwistor formalism, with the extensions of Penrose–Ferber relations, which relate $OSp(1|64)$ supertwistor space described by real coordinates ($\xi^2 = 0; R = 1 \ldots 65$)

$$T_R = (\lambda_r, \omega^r, \xi) , \quad (7)$$

[a]The relation (1) is the standard, minimal M-superalgebra. One can also add arbitrary spin-tensor central charges (see e.g. Refs. 5,6). The most general case was considered by Sezgin.[7]

with the generalized phase space (X_{rs}, P_{rs}) (see (1–2)). In Sect. 4 we shall describe algebraically $\frac{k}{32}BPS$ states by $n = 32 - k$ superspinors (7) representing $D = 11$ generalized supertwistors. These supertwistorial constituents we shall call BPS preons. It appears that our model geometrically corresponds to new type of Kaluza–Klein theory, with discrete internal extension of space-time coordinates.

2. $D = 11$ Conformal M-(Super)Algebra

Let us observe that the $D = 4$ conformal algebra $(P_\mu, M_{\mu\nu}, D, K_\mu)$ is endowed with the following three-grading structure

$$\begin{array}{ccccc} L_1 & & L_0 & & L_{-1} \\ P_\mu & & M_{\mu\nu}, D & & K_\mu \end{array} . \tag{8}$$

Grading (8) in determined by the scale dimensions of generators

$$[D, P_\mu] = P_\mu\,, \qquad [D, M_{\mu\nu}] = 0\,, \qquad [D, K_\mu] = -K_\mu \tag{9}$$

and it is easy to see that the conformal algebra (8) has two Poincaré subalgebras: $(P_\mu, M_{\mu\nu})$ and $(K_\mu, M_{\mu\nu})$. For $D = 4$ superconformal algebra $SU(2,2;1) = (P_{\mu\nu}$ $M_{\mu\nu}$, D, A, K_μ; Q_A, $Q_{\dot{A}}$, S_A, $S_{\dot{A}})$ the three-grading (8) is extended to the following five-grading

$$\begin{array}{ccccccc} L_1 & & L_{1/2} & & L_0 & L_{-1/2} & L_{-1} \\ P_\mu & & Q_A, \overline{Q}_{\dot{A}} & & M_{\mu\nu}, D, A & S_A, \overline{S}_{\dot{A}} & K_\mu \end{array} , \tag{10}$$

where consistently

$$\begin{aligned} [D, Q_A] &= \tfrac{1}{2} Q_A\,, & [D, S_A] &= -\tfrac{1}{2} S_A\,, \\ [D, \overline{Q}_{\dot{A}}] &= \tfrac{1}{2} Q_A\,, & [D, \overline{S}_{\dot{A}}] &= \tfrac{1}{2} \overline{S}_{\dot{A}}\,, \end{aligned} \tag{11}$$

and again $SU(2,2;1)$ contains as subsuperalgebras the Poincaré superalgebras $(P_\mu$, $M_{\mu\nu}$, Q_A; $\overline{Q}_{\dot{A}})$ and $(K_\mu, M_{\mu\nu}, S_A, S_{\dot{A}})$.

The structure of $D = 11$ generalized superconformal algebra, which we call conformal M-superalgebra is quite analogous. The $D = 11$ conformal M-algebra $Sp(64)$ can be in analogy to (8) described by the following three-grading

$$\begin{array}{ccccc} L_1 & & L_0 & & L_{-1} \\ Z_{rs} & & R_{rs} & & \widetilde{Z}_{rs} \\ 528 \text{ Abelian} & & GL(32;R) & & 528 \text{ Abelian} \\ \text{generators} & & \text{algebra} & & \text{generators} \end{array} . \tag{12}$$

We see that $Sp(64)$ contains two copies of generalized $D = 11$ Poincaré algebras, described by inhomogeneous $Sp(32)$ algebras $(Sp(32;R) \subset GL(32;R))$ with 528 Abelian translation generators.

The superextension of $D = 11$ conformal M-algebra $OSp(1;64)$ which we call conformal M-superalgebra is described by the following five-grading (see also Refs. 15,16)

$$\begin{array}{ccccccc} L_1 & L_{1/2} & L_0 & L_{-1/2} & L_{-1} \\ Z_{rs} & Q_r & R_{rs} & S_r & \widetilde{Z}_{rs} \end{array} , \tag{13}$$

where (Q_r, S_r) are the pair of 32-component supercharges, transforming as fundamental representations of $Sp(32)$, with $R_{rs} \subset Sp(32)$ if $R_{rs} = R_{sr}$. The subalgebras spanned by the generators (Q_r, Z_{rs}) and $(S_r, \widetilde{Z}_{rs})$ describe two copies of M-superalgebra given by the relations (1).

It should be added that the gradings (12,13) correspond to the grading structure of real Jordanian (super) algebras.[17,18]

3. $D = 11$ Supertwistors and Their Relation with Generalized Superspace

Let us recall two basic relations of Penrose twistor theory in $D = 4$[8,11]

(i) relation between the generators of Poincaré algebra and twistor components (2)

$$P_{A\dot{B}} = \lambda_A \lambda_B , \tag{14}$$

$$M_{AB} = \lambda_{(A} \overline{\omega}_{B)} , \qquad M_{\dot{A}\dot{B}} = \overline{\lambda}_{(\dot{A}} \omega_{\dot{B})} \tag{15}$$

where $M_{AB} = \frac{1}{2}(\sigma_{\mu\nu})_{AB} M^{\mu\nu}$ and $M_{\dot{A}\dot{B}} = \frac{1}{2}(\widetilde{\sigma}_{\mu\nu})_{\dot{A}\dot{B}} M^{\mu\nu}$[b]. The relations (15) can be extended to all 15 generators of $D = 4$ conformal algebra.

(ii) Penrose incidence relation between twistor and space-time coordinates

$$\omega^{\dot{A}} = i\lambda_B X^{B\dot{A}} \qquad \overline{\omega}^A = -iX^{A\dot{B}} \overline{\lambda}_B \tag{16}$$

where $X^{B\dot{A}} = (X^{A\dot{B}})^\star$ describe four real Minkowski coordinates if the $SU(2,2)$ twistor norm vanishes

$$(t,t) \equiv i\left(\lambda_A \overline{\omega}^A - \overline{\lambda}_{\dot{A}} \omega^{\dot{A}}\right) = 0 . \tag{17}$$

The relations (14–17) can be supersymmetrized. If we introduce the $D = 4$ supertwistor (t_α, η), which is the fundamental representation of $SU(2,2;1)$ with complex Grassmann variable η ($\eta^2 = \overline{\eta}^2 = 0$, $\{\eta, \overline{\eta}\} = 0$), the relations (14–15) has been extended by Ferber[19] to all generators of $D = 4$ superconformal group $SU(2,2;1)$.

The Penrose relations (16–17), firstly supersymmetrized in Ref. 19 look as follows

[b] We recall that $(\sigma_{\mu\nu})_{AB} = \frac{1}{2i}[(\sigma_\mu)_{A\dot{B}} \widetilde{\sigma}_\nu{}^{\dot{B}}{}_B - (\sigma_\nu)_{A\dot{B}} \widetilde{\sigma}_\mu{}^{\dot{B}}{}_B] = -\frac{i}{2}\epsilon_{\mu\nu\rho\tau}(\sigma^{\rho\tau})_{AB} = [(\widetilde{\sigma}_{\mu\nu})_{\dot{B}\dot{A}}]^\star$

$$\omega^{\dot{A}} = i\lambda_B \, Z^{B\dot{A}} \equiv i\lambda_B \left(X^{B\dot{A}} - i\theta^B \theta^{\dot{A}} \right)$$

$$\overline{\omega}^A = i \left(X^{A\dot{B}} + i\theta^A \theta^{\dot{B}} \right) \lambda_B$$

$$\eta = \lambda_A \theta^A \qquad \overline{\eta} = \overline{\lambda}_{\dot{A}} \overline{\theta}^{\dot{A}} \,. \tag{18}$$

For $D = 11$ the generalized twistors and supertwistors are real (see (7)) and the real $OSp(1;64)$ superalgebra $(R, S = 1 \ldots 64)$

$$\{Q_R, Q_S\} = R_{RS} \,, \tag{19}$$

can be obtained if we assume that[c].

$$R_{RS} = T_R \, T_S \qquad Q_R = \frac{1}{\sqrt{2}} T_R \xi \,, \tag{20}$$

where T_R describes $D = 11$ real twistorial quantum phase space ($\eta_{RS} = -\eta_{SR}$ is the $Sp(64)$ antisymmetric metric)

$$[T_R, T_S] = i\eta_{RS} \,, \tag{21}$$

supplemented with trivial one-dimensional Clifford algebra relation $\xi^2 = 1$.

The relations (19) are extended to $D = 11$ as follows:

$$\omega^r = (X^{rs} - i\theta^r \, \theta^s) \, \lambda_s \qquad \xi = \theta^r \, \lambda_r \,. \tag{22}$$

Relations (22) relate the $D = 11$ supertwistor space coordinates (7) with the extended $D = 11$ superspace (X_{rs}, θ_s), described by 528 bosonic and 32 fermionic coordinates.

4. BPS States in M-Theory and Composites of BPS Preons

The $\frac{k}{32}BPS$ state $|k\rangle$ can be defined as an eigenstate of generalized momenta generators

$$Z_{rs}|k\rangle = z_{rs}|k\rangle \,, \tag{23}$$

such that $\det z_{rs} = 0$. The number k determines the rank of generalized momenta matrix z_{rs}

$$\frac{k}{32}\text{BPS state: } \{\text{rank } z_{rs} = n = 32 - k; \quad 1 \leq k < 32\} \,. \tag{24}$$

¿From (24) follows that the BPS state $|k\rangle$ preserves a fraction $\nu = \frac{k}{32}$ of supersymmetries.

[c]By Bott periodicity this realization is related with twistor framework in $D = 3$ (see Ref. 20), also with real structure. In $D = 5, 6, 7$ one has to use the extension of Penrose framework to quaternionic twistors (see e.g. Ref. 21 for $D = 6$).

We call BPS preon the hypothetical primary object carrying the following generalized momenta[1]

$$Z_{rs} = \lambda_r \lambda_s \,. \tag{25}$$

The formula (25) corresponds to putting $n = 1$ in the relation (5) and describes $\frac{31}{32}BPS$ state. More general formula (5) describes the generalized momenta of a system composed out of n BPS preons and it describes (for $1 \leq n \leq 32$) the $\frac{k}{32}BPS$ state (we recall that $k = 32 - n$).

The number $n = 32 - k$ of zero eigenvalues of the matrix z_{rs} determines the number of independent supercharges $Q_r^{(i)}$, annihilating the BPS state $|k\rangle$. These supersymmetries, preserving the BPS state, are called in p-brane theory the κ-transformations. We see that the supersymmetric $D = 11$ single BPS preon dynamics should have 31 κ-symmetries. Recently[22] such dynamical superparticle models[d] with fundamental $OSp(1;2n)$ superspinor as basic variable has been proposed. It should be recalled here (see e.g. Ref. 24) that in the standard super p-brane formulations half of the supersymmetries are promoted to κ-transformations, i.e. in $D = 11$ we obtain 16 κ-transformations.

Using the $D = 11$ supertwistor description with the relations (22) and (25) providing a bridge between BPS preons and generalized space-time, we can formulate three different geometric pictures:

(i) Purely supertwistorial picture, with basic phase space parametrized by BPS preon coordinates $T_R^{(i)}$ (see (7)). The canonical Liouville one-form describing free action is given by the relation

$$\Omega_1 = \sum_{i=1}^{n} \left(\omega^{(i)r}\, d\lambda_r^{(i)} + \xi^{(i)}\, d\xi^{(i)} \right) , \tag{26}$$

which can be supplemented by some algebraic constraints.

(ii) Mixed geometric picture, with the components $\omega^{(i)}$ expressed by means of the relation (22). One obtains from (26)

$$\Omega_2 = \sum_{i=1}^{n} \lambda_r^{(i)r}\, \lambda_s^{(i)r}\, (dX^{rs} - i\theta^r\, d\theta^s) \,, \tag{27}$$

(iii) Generalized space-time picture, with the relation (5) inserted in (27).

$$\Omega_3 = Z_{rs}\, (dX^{rs} - i\theta^r\, d\theta^s) \,. \tag{28}$$

The application of these three geometric pictures to the description of $D = 11$ dynamics (for $n > 1$) is under consideration.

[d] For $D = 4, 6$ and 10 see Refs. 22, 23.

5. Final Remarks

We mention here two interesting aspects of the presented approach which deserve further attention;

(i) geometric confinement of BPS preons
Because the space-time coordinates are composed out of preonic degrees of freedom, the $D = 11$ space-time point can be determined only in terms of at least 16 preonic set of spinorial coordinates. This is the $D = 11$ extension of known property of Penrose theory in four dimensions with two twistors needed for the definition of composite Minkowski space-time points.

(ii) internal symmetries
The formula (5) expresses 528 generalized momenta in terms of $32n$ preonic spinorial coordinates $\lambda_r^{(i)}$ $(i = 1, \ldots n)$. The internal symmetries can be obtained by interchanging BPS preons. For the case n=16 corresponding to the choice of $\nu = \frac{1}{2}$SUSY one can introduce internal $O(16)$ symmetries, leaving the values of Z_{rs} invariant.

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GAUSSIAN ORTHOGONAL ENSEMBLE FOR THE LEVEL SPACING STATISTICS OF THE QUANTUM FOUR-STATE CHIRAL POTTS MODEL

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We have performed a Random Matrix Theory (RMT) analysis of the quantum four state chiral Potts chain for different sizes of the quantum chain up to eight sites, and for different unfolding methods. Our analysis shows that one generically has a Gaussian Orthogonal Ensemble statistics for the unfolded spectrum instead of the GUE statistics one could expect. Furthermore a change from the generic GOE distribution to a Poisson distribution occurs when the hamiltonian becomes integrable. Therefore, the RMT analysis can be seen as a detector of "higher genus integrability".

Introduction : the quantum chiral Potts chain

Since the pioneering work of Wigner[1] and Dyson,[2] Random Matrix Theory (RMT) has been applied successfully in various domains of physics. One motivation is to describe, in a united universal framework, various phenomena implying chaos[3] or at least complexity. An extreme case is the emergence of integrability which manifests itself in the drastic change of the generic wignerian energy level spacing distribution into poissonian distribution. The first examples of this connection emerged when one considered simple harmonic oscillators or free fermions models. This reduction to Poisson distribution reflects nothing but the *independence* of the eigenvalues. At this point it is natural to ask whether this link between Poisson reduction and Yang–Baxter integrability still holds when the solutions of the Yang–Baxter equations are no longer parametrized in terms of abelian varieties. The perfect example to address this question is the chiral Potts model for which Au-Yang et-al have found a *higher genus* Yang–Baxter solution.[4] The Hamiltonian of the quantum chiral Potts chain first introduced by Howes, Kadanoff and den Nijs[5] and also by von Gehlen and

Rittenberg[6] is defined as :

$$H \equiv H_X + H_{ZZ} = \sum_j H_{jj+1} = -\sum_j \sum_{n=1}^{N-1} [\overline{\alpha_n} \cdot (X_j)^n + \alpha_n \cdot (Z_j Z_{j+1}^\dagger)^n] \qquad (1)$$

where $X_j = I_N \otimes \cdots \otimes X \otimes \cdots \otimes I_N$ and $Z_j = I_N \otimes \cdots \otimes Z \otimes \cdots \otimes I_N$. Here I_N is $N \times N$ unit matrix, while X and Z are $N \times N$ matrices whose elements are defined by $Z_{j,m} = \delta_{j,m} \exp[2\pi i(j-1)/N]$ and $X_{j,m} = \delta_{j,m+1} \pmod N$. The self-dual model[7] corresponds to $\alpha_n = \overline{\alpha_n}$. Some spectral analysis of this model have been performed for the quantum self-dual model or the 3-state model.[6,8] In this paper we examine the $N = 4$ (four state chiral Potts model) non self-dual case. The conditions for the quantum Hamiltonian to commute with the transfer matrix family (integrability conditions) read (see equations (33a), (33b), (33c) and (33d) in Ref. 9) :

$$\frac{\overline{\alpha_2}^2}{\overline{\alpha_1}\overline{\alpha_3}} = \frac{\alpha_2^2}{\alpha_1\alpha_3}, \qquad \frac{\overline{\alpha_1}^2 + \overline{\alpha_3}^2}{\overline{\alpha_2}} = \frac{\alpha_1^2 + \alpha_3^2}{\alpha_2} \qquad (2)$$
$$(\alpha_1^2 - \alpha_3^2)(2\alpha_2^2 - \alpha_1\alpha_3) = 0, \qquad (\overline{\alpha_1}^2 - \overline{\alpha_3}^2)(2\overline{\alpha_2}^2 - \overline{\alpha_1}\overline{\alpha_3}) = 0$$

In order to have a *real spectrum* we also choose to have an hermitian hamiltonian restricting to conditions $\alpha_1 = \alpha_3^\star$ and $\overline{\alpha_1} = \overline{\alpha_3}^\star$ (where the star denotes the complex conjugate). A possible parametrization is then :

$$\alpha_1 = \alpha_3^\star = \sqrt{1+r} + i\sqrt{1-r}, \qquad \alpha_2 = 1 \qquad (3)$$
$$\overline{\alpha_1} = \overline{\alpha_3}^\star = \sqrt{n^2+rn} + i\sqrt{n^2-rn}, \qquad \overline{\alpha_2} = n$$

where r and n are real. The value $n = 1$ corresponds to the self dual situation.

1. The RMT machinery.

RMT analysis considers the spectrum of the (quantum) Hamiltonian, or of the transfer matrix, as a collection of numbers, and looks for some possibly universal statistical properties of this collection of numbers. Obviously, neither the raw spectrum, nor the raw level spacing distribution, have any universal properties. In order to uncover universal properties, one has to perform some normalization of the spectrum: the so-called *unfolding* operation. This amounts to making the *local* density of eigenvalues of the spectrum equal to unity everywhere[a]. In other words, one subtracts the regular part from the integrated density of states: one considers only the fluctuations. It has been found that the unfolded spectra of many quantum systems are very close to one of four archetypal situations described by four statistical ensembles. For integrable models this is the statistical ensemble of diagonal random

[a]The unfolding can be achieved by different means. Let us note however that there is no rigorous prescription and the "best criterion" is the insensitivity of the final result to the method employed or to the parameters (for "reasonable" variation). A detailed explanation and tests of these methods of unfolding are given in Ref. 10.

matrices and the level spacing distribution is close to a Poissonian (exponential) distribution, $P(s) = \exp(-s)$. For non-integrable systems it can be the Gaussian Orthogonal Ensemble (GOE), the Gaussian Unitary Ensemble (GUE), or the Gaussian Symplectic Ensemble (GSE), depending on the symmetries of the model under consideration. If the hamiltonian is time reversal invariant[13] the level spacing distribution is either described by the Gaussian Orthogonal Ensemble (GOE), or by the Gaussian Symplectic Ensemble (GSE):

$$P_{\mathrm{GOE}}(s) = \frac{\pi}{2} s \exp(-\pi s^2/4), \qquad P_{\mathrm{GSE}}(s) = B^3 s^4 \exp(-B s^2) \tag{4}$$

where $B = \left(\frac{8}{3}\right)^2 \frac{1}{\pi} \simeq 2.263$. Note that GOE can also occurs in a slightly more general framework ("false" time-reversal violation, A-adapted basis[12]). When one does not have any time-reversal symmetry (or "false time-reversal symmetry") the Gaussian Unitary Ensemble distribution should appear :

$$P_{\mathrm{GUE}}(s) = \frac{32}{\pi^2} s^2 \exp(-4s^2/\pi) \tag{5}$$

To quantify the "degree" of level repulsion, it may be convenient to use a parametrized distribution which interpolates between the Poisson law and the GOE Wigner law. Among the many possible distributions we have chosen the *Brody distribution*:

$$P_\beta(s) = (1+\beta)\, c_2\, s^\beta \exp(-c_2 s^{\beta+1}), \quad \text{with} \quad c_2 = \left[\Gamma\left(\frac{\beta+2}{\beta+1}\right)\right]^{1+\beta} \tag{6}$$

1.1. *Representation theory*

In the presence of symmetries, one should distinguish eigenstates according to their quantum numbers. This is an *essential requirement of the method.* For instance both lattice shift and shift of colour commute with the hamiltonian H. They generate a symmetry group $S = Z_L \otimes Z_4$ which does not depend on the parameters $\alpha_i, \overline{\alpha_i}$ of the hamiltonian H. Since the group $S = Z_L \otimes Z_4$ is abelian one may diagonalize simultaneously all the elements of the group S as well as the hamiltonian H on the S-invariant spaces. This amounts to block-diagonalizing H and to split the spectrum of H into the many spectra of each block. The construction of the projectors is done with the help of the character table of irreducible representations of the symmetry group. Details can be found in Ref. 10 and Ref. 14.

In this work we concentrate on the four-state case ($N = 4$) of the quantum hamiltonian (1). For generic r and n in parametrization Eq. (3), the total symmetry group is $Z_L \otimes Z_4$. Since the characters of $Z_L \otimes Z_4$ are complex, one has to use complex numbers even though the final results are real, which increases the programming difficulties. We always restricted ourselves to hermitian hamiltonians. Consequently the blocks are also hermitian and there are only *real* eigenvalues. The diagonalization is performed using standard methods of linear algebra (contained in the LAPACK library).

2. Results.

We show in this paper that the RMT analysis can act as an integrability detector. More specifically we want to exhibit the transition to integrability when the parameters meet conditions Eq.(2). We thus choose to move in the $\alpha_i, \overline{\alpha_i}$ parameter space along a trajectory compatible with the hermiticity of the hamiltonian, generalizing the parametrization (4) and crossing the integrable variety Eq.(2) :

$$\begin{aligned} \alpha_1 &= \alpha_3^\star = \sqrt{1+r} + i\sqrt{1-r}\,, & \alpha_2 &= t \\ \overline{\alpha_1} &= \overline{\alpha_3}^\star = \sqrt{n^2+rn} + i\sqrt{n^2-rn}\,, & \overline{\alpha_2} &= n \end{aligned} \tag{7}$$

where t, r and n are real, with $n \neq 1$ to avoid the self-dual case. The value $\alpha_2 = t = 1$ thus corresponds to the occurrence of genuinely "higher genus integrability" on this trajectory.

We have constructed the quantum Hamiltonian (1), of the four state Potts model (1), for various linear sizes, up to size eight ($L = 8$), leading to matrices of size up to $4^8 \times 4^8 = 65536 \times 65536$. The results, displayed below, show that the size $L = 8$ is sufficient to answer the question we addressed. Using the complex characters and projectors associated with the group $Z_L \otimes Z_4$ we have performed the block diagonalization of the hamiltonian. The sum of the dimensions of all the blocks corresponding to the $8 \times 4 = 32$ representations, is $4^8 = 65536$ as it should. We then performed the unfolding in each block independently. The behavior in the

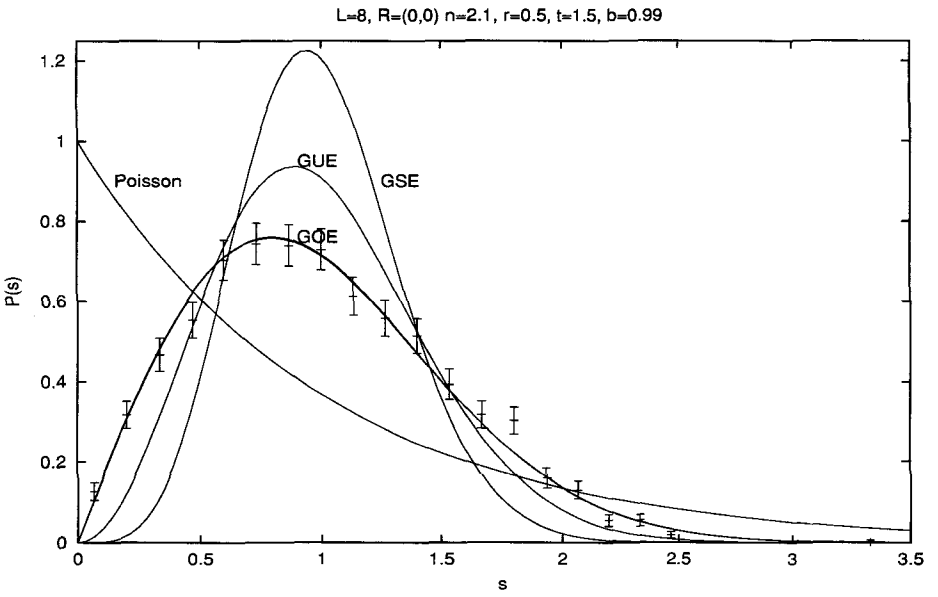

Fig. 1. Level spacing distribution versus GOE, GUE, GSE and Poisson.

various blocks (representations) is not significantly different. We also compared four different unfolding procedures, again getting similar results. We display the results on the largest size $L = 8$ for the best unfolding procedure, namely the gaussian unfolding.

Figure 1 shows the level spacing distribution $P(s)$, for the representation (0,0) and for $r = 0.5$, $n = 2.1$, and $t = 1.5$, which corresponds to $\alpha_1 = \alpha_3^\star = 1.225 + i\,0.707$, $\alpha_2 = t = 1.5$, $\overline{\alpha_1} = \overline{\alpha_3}^\star = 2.337 + i\,1.833$ and $\overline{\alpha_2} = 2.1$.

This figure shows the energy level spacing distribution and the corresponding brody fit (6) for the (least square) best value found to be $\beta_{brody} = 0.99$. On the same figure the GOE level spacing distribution is also displayed, both curves are almost indistinguishable. The GUE or GSE level spacing distribution are *clearly ruled out*, as well, of course, as the Poisson distribution. Very similar results are obtained for all the distributions corresponding to the other representations and other values away from the integrability value $\alpha_2 = t = 1$.

Let us now consider the (higher genus) integrable case which corresponds, with our parametrization, to $\alpha_2 = t = 1$.

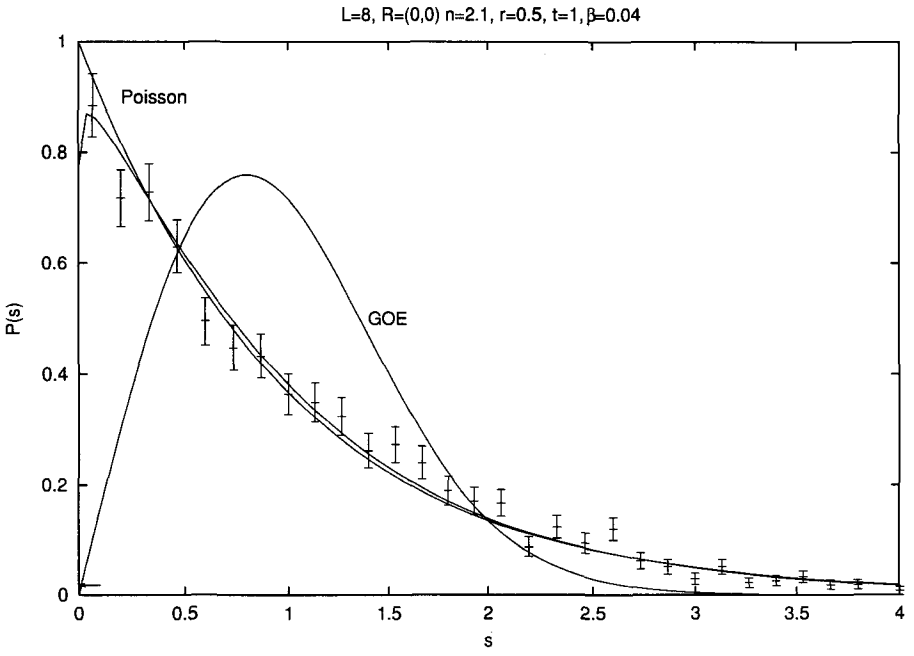

Fig. 2. Level spacing distribution on the integrability variety.

Figure 2 displays the level spacing distribution, compared to a Poisson distribution (and also to the GOE level spacing distribution), for the integrable case

$r = 0.5$, $n = 2.1$, and $t = 1$ which corresponds to $\alpha_1 = \alpha_3^\star = 1.225 + i\ 0.707$, $\alpha_2 = t = 1$, $\overline{\alpha_1} = \overline{\alpha_3}^\star = 2.337 + i\ 1.833$ and $\overline{\alpha_2} = 2.1$. The best brody distribution approximation of the data is found to be for $\beta_{brody} = 0.04$ using a least square fit. We have obtained very similar results (namely an extremely good agreement with a Poisson distribution) with other values of the parameters n and r, and for the various representations, when t is kept equal to the (higher genus) integrability value $t = 1$.

The RMT analysis can therefore be used to *detect integrability even when the integrability is not associated with abelian curves but is a more subtle integrability where higher genus curves occur.*

This extremely good agreement with an *independent eigenvalues* framework is found for $t = 1$ exactly. When t is slightly different from 1, one is clearly no longer Poissonian in agreement with the fact that the Poissonian framework should only correspond to the integrable value $t = 1$. In order to quantify the (finite size) transition from integrability to chaos, we calculate the best brody parameter, as a function of the parameter t, keeping r and n constant. Figure 3 displays β_{brody}, as a function of t, for all the representations.

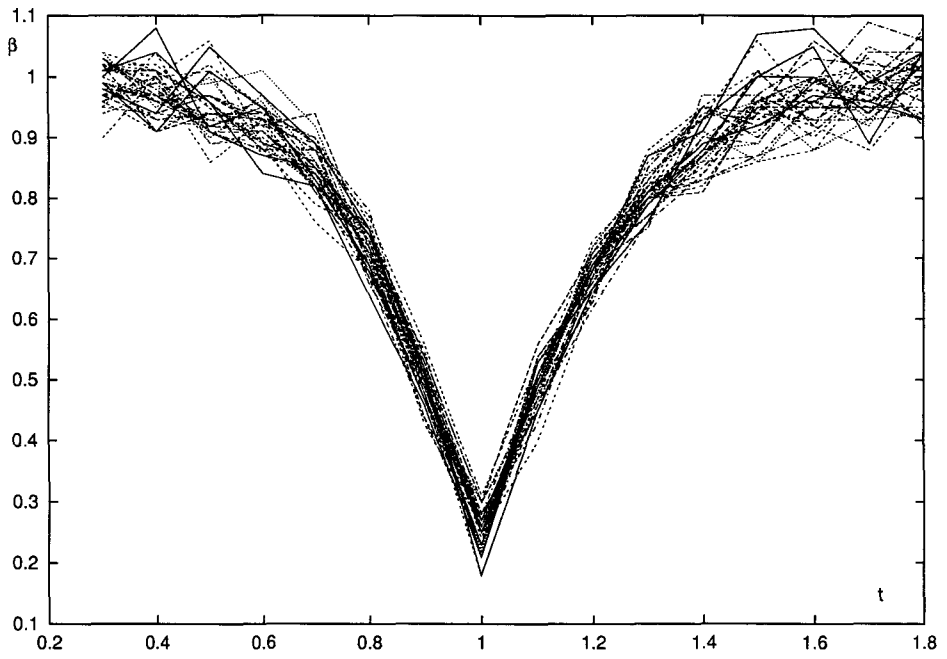

Fig. 3. The brody parameter, as a function of the parameter t.

These results confirm a quite sharp transition from a GOE distribution to a Poisson distribution. In the thermodynamic limit one can expect β_{brody} to be equal

to the GOE value $\beta_{brody} = 1$ for every value of the parameter t, except at point $t = 1$, where the Poisson value $\beta_{brody} = 0$ should occur.

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COMMENTS ON THE DEFORMED W_N ALGEBRA*

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We obtain an explicit expression for the defining relation of the deformed W_N algebra, $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$. Using this expression we can show that, in the $q \to 1$ limit, $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ with $t = e^{-\frac{2\pi i}{N}} q^{\frac{k+N}{N}}$ reduces to the $\mathfrak{sl}_N$-version of the Lepowsky–Wilson's $\mathcal{Z}$-algebra of level k, $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$. In other words $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ with $t = e^{-\frac{2\pi i}{N}} q^{\frac{k+N}{N}}$ can be considered as a q-deformation of $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$.

In the appendix given by H. Awata, S. Odake and J. Shiraishi, we present an interesting relation between $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ and ζ-function regularization.

1. Introduction

One of our motivation for study of elliptic algebras (deformed Virasoro and W algebras, elliptic quantum groups, etc.) is to clarify the symmetry of massive integrable models. Massive integrable models includes quantum field theories with mass scale and solvable statistical lattice models. Typical examples of the latter are models based on $\mathfrak{sl}_2$: Andrews–Baxter–Forester (ABF) model and Baxter's eight vertex model. About these models we know the following:[1]

model	ABF(III)	8 vertex
Boltzmann weight	face type	vertex type
algebra	$\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ $(\mathcal{B} \otimes \{P, e^Q\} = U_{q,p}(\widehat{\mathfrak{sl}}_2))$	$\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$
gradation (energy level of H_C)	homogeneous gradation	principal gradation
space of states	irr. rep. space of DVA	irr. rep. space of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$
free field realization	direct (construction of VO)	indirect (map to ABF)

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In order to obtain more direct free field realization of the eight vertex model and its higher rank generalization, it may be useful to study (deformed) current algebras of $\mathfrak{sl}_N$ in principal gradation. Motivated by this, Hara *et al.*[2] studied free field realization of the Lepowsky–Wilson's $\mathcal{Z}$-algebra[3] and found some relation between the deformed Virasoro algebra (DVA) and $\mathcal{Z}$-algebra. Recently Shiraishi constructed a direct free field realization of the eight vertex model with a specific parameter $p = q^3$, where the type II vertex operator is given by the DVA current.[4]

In this article we extend the relation between DVA and $\mathcal{Z}$-algebra to the higher rank case. In section 2 we present an explicit expression for the defining relation of the deformed W_N algebra. This is a main result of this article. In section 3, by using this explicit expression, we show that the deformed W_N algebra reduces to the $\mathfrak{sl}_N$-version of the Lepowsky–Wilson's $\mathcal{Z}$-algebra in some limit. In the appendix given by Awata, Odake and Shiraishi, we present an interesting relation between the deformed W_N algebra and ζ-function regularization.[5]

2. Deformed W_N Algebra

2.1. *Definition*

Let us recall the definition of the deformed W_N algebra, $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$.[6,7] It is defined through a free field realization. This algebra has two parameters(q and t), and we set $t = q^\beta$ and $p = qt^{-1}$. Let us introduce fundamental bosons h_n^i ($n \in \mathbb{Z}$; $i = 1, \cdots, N$; $\sum_{i=1}^N p^{in} h_n^i = 0$) which satisfy

$$[h_n^i, h_m^j] = -\frac{1}{n}(1-q^n)(1-t^{-n})\frac{1-p^{(N\delta_{i,j}-1)n}}{1-p^{Nn}}p^{Nn\theta(i<j)}\delta_{n+m,0}, \tag{1}$$

where $\theta(P) = 1$ or 0 if the proposition P is true or false, respectively. Exponentiated boson $\Lambda_i(z)$ ($i = 1, \cdots, N$) is defined by

$$\Lambda_i(z) = :\exp\Big(\sum_{n\neq 0} h_n^i z^{-n}\Big): q^{\sqrt{\beta}h_0^i} p^{\frac{N+1}{2}-i}. \tag{2}$$

Here $:*:$ stands for the usual normal ordering for bosons, *i.e.*, h_n^i with $n \geq 0$ are in the right. By using this $\Lambda_i(z)$, $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ current $W^i(z) = \sum_{n\in\mathbb{Z}} W_n^i z^{-n}$ ($i = 1, \cdots, N-1$) is given by

$$W^i(z) = \sum_{1\leq j_1<j_2<\cdots<j_i\leq N} :\Lambda_{j_1}(p^{\frac{i-1}{2}}z)\Lambda_{j_2}(p^{\frac{i-3}{2}}z)\cdots\Lambda_{j_i}(p^{-\frac{i-1}{2}}z):, \tag{3}$$

and we set $W^0(z) = W^N(z) = 1$. (Remark that $\Lambda_i(z)$ corresponds to the weight of vector representation of $\mathfrak{sl}_N$ and $W^i(z)$ corresponds to the i-th rank antisymmetric tensor representation.) $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ is defined as an associative algebra over $\mathbb{C}$ generated by W_n^i.[a]

[a]It is also defined as a commutant of the screening currents.[6,8]

The highest weight state $|\lambda\rangle$ is characterized by $W_n^i|\lambda\rangle = 0$ $(n > 0)$ and $W_0^i|\lambda\rangle = w^i(\lambda)|\lambda\rangle$ $(w^i(\lambda) \in \mathbb{C})$, and the highest weight representation space is obtained by successive action of W_{-n}^i $(n > 0)$.

Since $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ has two parameters(q and t), we can take its various limit by relating q and t. In the following limit[b]

$$\text{Limit I}: \begin{cases} q = e^{\hbar}, & \hbar \to 0 \\ t = q^{\beta}, & \beta : \text{fixed} \quad (\alpha_0 = \sqrt{\beta} - \frac{1}{\sqrt{\beta}}), \end{cases} \tag{4}$$

$\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ reduces to the W_N algebra with the Virasoro central charge $c = (N-1)(1-N(N+1)\alpha_0^2)$ because the q-Miura transformation of $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)$ becomes the Miura transformation of W_N algebra. Each DWA current $W^i(z)$, however, reduces to some linear combination of W_N currents.

2.2. *Relation*

In order to write down relations between DWA currents, we define the delta function $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ and the structure function $f^{i,j}(z) = \sum_{\ell\geq 0} f_\ell^{i,j} z^\ell$ $(1 \leq i, j \leq N-1)$,

$$f^{i,j}(z) = \exp\left(\sum_{n>0} \frac{1}{n}(1-q^n)(1-t^{-n})\frac{1-p^{\min(i,j)n}}{1-p^n}\frac{1-p^{(N-\max(i,j))n}}{1-p^{Nn}}p^{\frac{|i-j|}{2}n}z^n\right). \tag{5}$$

It has been expected that DWA currents satisfy quadratic relations, $f^{i,j}(\frac{z_2}{z_1})W^i(z_1)W^j(z_2) - W^j(z_2)W^i(z_1)f^{j,i}(\frac{z_1}{z_2}) =$ (terms containing delta function), in mode expansion it becomes

$$\begin{aligned}[W_n^i, W_m^j] = &-\sum_{\ell\geq 1} f_\ell^{i,j}\left(W_{n-\ell}^i W_{m+\ell}^j - W_{m-\ell}^j W_{n+\ell}^i\right) \\ &+(\text{contribution from the terms containing delta function}).\end{aligned} \tag{6}$$

For $i = 1$ and $j \geq 1$ case, the relation is[6,7]

$$\begin{aligned}&f^{1,j}(\tfrac{z_2}{z_1})W^1(z_1)W^j(z_2) - W^j(z_2)W^1(z_1)f^{j,1}(\tfrac{z_1}{z_2}) \qquad (j \geq 1) \\ &= -\frac{(1-q)(1-t^{-1})}{1-p}\left(\delta(p^{\frac{j+1}{2}}\tfrac{z_2}{z_1})W^{j+1}(p^{\frac{1}{2}}z_2) - \delta(p^{-\frac{j+1}{2}}\tfrac{z_2}{z_1})W^{j+1}(p^{-\frac{1}{2}}z_2)\right),\end{aligned} \tag{7}$$

and for $i = 2$ and $j \geq 2$ case, the relation is[7]

$$\begin{aligned}&f^{2,j}(\tfrac{z_2}{z_1})W^2(z_1)W^j(z_2) - W^j(z_2)W^2(z_1)f^{j,2}(\tfrac{z_1}{z_2}) \qquad (j \geq 2) \\ &= -\frac{(1-q)(1-t^{-1})}{1-p}\frac{(1-qp)(1-t^{-1}p)}{(1-p)(1-p^2)} \\ &\qquad \times\left(\delta(p^{\frac{j}{2}+1}\tfrac{z_2}{z_1})W^{j+2}(pz_2) - \delta(p^{-\frac{j}{2}-1}\tfrac{z_2}{z_1})W^{j+2}(p^{-1}z_2)\right) \\ &-\frac{(1-q)(1-t^{-1})}{1-p}\left(\delta(p^{\frac{j}{2}}\tfrac{z_2}{z_1})\,{}^{\circ}_{\circ}W^1(p^{-\frac{1}{2}}z_1)W^{j+1}(p^{\frac{1}{2}}z_2)\,{}^{\circ}_{\circ}\right.\end{aligned}$$

[b]Usually we call this limit as a conformal limit. However there are many other limits in which the resultant algebras are related to conformal field theory.

$$
\begin{aligned}
&\qquad -\delta(p^{-\frac{j}{2}}\tfrac{z_2}{z_1})^{\circ}_{\circ}W^1(p^{\frac{1}{2}}z_1)W^{j+1}(p^{-\frac{1}{2}}z_2)^{\circ}_{\circ}\Big)\\
&+\frac{(1-q)^2(1-t^{-1})^2}{(1-p)^2}\Big(\delta(p^{\frac{j}{2}}\tfrac{z_2}{z_1})\Big(\frac{p^2}{1-p^2}W^{j+2}(pz_2)+\frac{1}{1-p^j}W^{j+2}(z_2)\Big) \qquad (8)\\
&\qquad -\delta(p^{-\frac{j}{2}}\tfrac{z_2}{z_1})\Big(\frac{p^j}{1-p^j}W^{j+2}(z_2)+\frac{1}{1-p^2}W^{j+2}(p^{-1}z_2)\Big)\Big).
\end{aligned}
$$

Here a normal ordering for currents $^{\circ}_{\circ} * ^{\circ}_{\circ}$ is defined by

$$
\begin{aligned}
&{}^{\circ}_{\circ}W^i(rz)W^j(z)^{\circ}_{\circ}\\
&=\oint\frac{dz'}{2\pi iz'}\Big(\frac{1}{1-\frac{rz}{z'}}f^{i,j}(\tfrac{z}{z'})W^i(z')W^j(z)+\frac{\frac{z'}{rz}}{1-\frac{z'}{rz}}W^j(z)W^i(z')f^{j,i}(\tfrac{z'}{z})\Big)\\
&=\sum_{n\in\mathbb{Z}}\sum_{m=0}^{\infty}\sum_{\ell=0}^{m}f_\ell^{i,j}\Big(r^{m-\ell}W^i_{-m}W^j_{n+m}+r^{\ell-m-1}W^j_{n-m-1}W^i_{m+1}\Big)\cdot z^{-n}, \qquad (9)
\end{aligned}
$$

where $\frac{1}{1-z}$ stands for $\sum_{n\geq 0}z^n$. Due to this normal ordering, infinite sums in the RHS of (6) become finite sums on the highest weight representation space.

Eqs.(7) and (8) are directly calculated by using the commutation relation of h^i_n. In principle, we can continue this calculation for $i\geq 3$ cases, but in practice it is hopeless. So we use another method: fusion and induction. To write down general formula, we extend the range$(0\leq i\leq N)$ of $W^i(z)$ and that$(1\leq i,j\leq N-1)$ of $f^{i,j}(z)$ to $i\in\mathbb{Z}$ and $i,j\in\mathbb{Z}$ respectively ; $W^i(z)=0$ for $i<0$ or $i>N$, and $f^{i,j}(z)$ is given by (5) for all $i,j\in\mathbb{Z}$.

Explicit expression of the defining relation of DWA$(\widehat{\mathfrak{sl}}_N)_{q,t}$ is as follows:

$$
\begin{aligned}
&f^{i,j}(\tfrac{z_2}{z_1})W^i(z_1)W^j(z_2)-W^j(z_2)W^i(z_1)f^{j,i}(\tfrac{z_1}{z_2}) \qquad (0\leq i\leq j\leq N)\\
&=-\frac{(1-q)(1-t^{-1})}{1-p}\sum_{k=1}^{i}\prod_{l=1}^{k-1}\gamma(p^{l+\frac{1}{2}})\\
&\quad\times\Big(\delta(p^{\frac{j-i}{2}+k}\tfrac{z_2}{z_1})f^{i-k,j+k}(p^{-\frac{j-i}{2}})W^{i-k}(p^{-\frac{k}{2}}z_1)W^{j+k}(p^{\frac{k}{2}}z_2)\\
&\qquad-\delta(p^{-(\frac{j-i}{2}+k)}\tfrac{z_2}{z_1})f^{i-k,j+k}(p^{\frac{j-i}{2}})W^{i-k}(p^{\frac{k}{2}}z_1)W^{j+k}(p^{-\frac{k}{2}}z_2)\Big), \qquad (10)
\end{aligned}
$$

where $\gamma(p^{\frac{1}{2}}z)=\dfrac{(1-qz)(1-t^{-1}z)}{(1-z)(1-pz)}$. We can rewrite the RHS of this relation in terms of the normal ordering $^{\circ}_{\circ} * ^{\circ}_{\circ}$ by repeated use of the following formula, which is obtained from (9) and (10),

$$
\begin{aligned}
&f^{i,j}(r^{-1})W^i(rz)W^j(z) \qquad (0\leq i\leq j\leq N)\\
&={}^{\circ}_{\circ}W^i(rz)W^j(z)^{\circ}_{\circ}+\frac{(1-q)(1-t^{-1})}{1-p}\sum_{k=1}^{i}\prod_{l=1}^{k-1}\gamma(p^{l+\frac{1}{2}})\\
&\quad\times\Big(\frac{1}{1-rp^{-(\frac{j-i}{2}+k)}}f^{i-k,j+k}(p^{-\frac{j-i}{2}})W^{i-k}(p^{\frac{j-i+k}{2}}z)W^{j+k}(p^{\frac{k}{2}}z)\\
&\qquad-\frac{1}{1-rp^{\frac{j-i}{2}+k}}f^{i-k,j+k}(p^{\frac{j-i}{2}})W^{i-k}(p^{-\frac{j-i+k}{2}}z)W^{j+k}(p^{-\frac{k}{2}}z)\Big), \qquad (11)
\end{aligned}
$$

where $r \in \mathbb{C}$ is a "good" number (such that it does not give poles, see (15)). For example, (8) is easily recovered by (10) with $i = 2$ and (11) with $i = 1$.

In order to prove (10) we present some formulas. Direct calculation shows

$$f^{1,j}(p^{\pm\frac{i+1}{2}}z)f^{i,j}(z) = f^{i+1,j}(p^{\pm\frac{1}{2}}z) \times \begin{cases} 1 & i < j \\ \gamma(p^{\pm\frac{i-j+1}{2}}z) & i \geq j \end{cases} \quad (j \geq 1), \tag{12}$$

$$f^{1,i}(p^{\pm(\frac{j-i}{2}+k)}z)f^{1,j}(z) = f^{1,i-k}(p^{\pm\frac{j-i+k}{2}}z)f^{1,j+k}(p^{\pm\frac{k}{2}}z) \quad (i,j,i-k,j+k \geq 1), \tag{13}$$

$$f^{1,i}(p^{\pm\frac{j+i}{2}}z)f^{1,j}(z) = f^{1,j+i}(p^{\pm\frac{i}{2}}z)\gamma(p^{\pm\frac{j}{2}}z) \quad (i,j \geq 1), \tag{14}$$

and $f^{a,b}$ in the RHS of (10) is regular. By computing $\langle\lambda|f^{i,j}(\frac{z_2}{z_1})W^i(z_1)W^j(z_2)|\lambda\rangle$ in the free field realization, we can show that (10) implies

$$\text{Poles of } f^{i,j}(\tfrac{z_2}{z_1})W^i(z_1)W^j(z_2) \ (0 \leq i \leq j \leq N) \text{ are}$$
$$\frac{z_2}{z_1} = p^{\pm(\frac{j-i}{2}+k)} \ (1 \leq k \leq \min(i, N-j)), \tag{15}$$

because $\langle\lambda|f^{i,j}(\frac{z_2}{z_1})W^i(z_1)W^j(z_2)|\lambda\rangle$ is a Taylor series in $\frac{z_2}{z_1}$, and for any states of the highest weight representation space, $|\psi\rangle$ and $|\phi\rangle$, $\langle\psi|f^{i,j}(\frac{z_2}{z_1})W^i(z_1)W^j(z_2)|\phi\rangle$ differs from $\langle\lambda|f^{i,j}(\frac{z_2}{z_1})W^i(z_1)W^j(z_2)|\lambda\rangle$ only for finite number of terms (Laurent polynomials in z_1 and z_2), which do not create other poles. (See also Appendix C of Ref. 9 where different notation is used.) Therefore $f^{a,b}W^aW^b$ in the RHS of (10) is regular and we can reverse its order, $f^{a,b}(p^c)W^aW^b = f^{b,a}(p^{-c})W^bW^a$. From (7) (or by using the free field realization), we have the following fusion relation

$$\lim_{z_1 \to p^{\pm\frac{j+1}{2}}z_2} \left(1 - p^{\pm\frac{j+1}{2}}\frac{z_2}{z_1}\right) f^{1,j}(\tfrac{z_2}{z_1})W^1(z_1)W^j(z_2)$$
$$= \mp\frac{(1-q)(1-t^{-1})}{1-p}W^{j+1}(p^{\pm\frac{1}{2}}z_2) \quad (1 \leq j \leq N), \tag{16}$$

and if (10) is correct, (10) implies

$$\lim_{z_2 \to p^{\mp\frac{j+i}{2}}z_1} \left(1 - p^{\mp\frac{j+i}{2}}\frac{z_1}{z_2}\right) f^{j,i}(\tfrac{z_1}{z_2})W^j(z_2)W^i(z_1)$$
$$= \pm\frac{(1-q)(1-t^{-1})}{1-p}\prod_{l=1}^{i-1}\gamma(p^{l+\frac{1}{2}}) \cdot W^{j+i}(p^{\mp\frac{j}{2}}z_1) \quad (0 \leq i \leq j \leq N). \tag{17}$$

Proof of (10) : (i) The case $i = 0$ and $i \leq \forall j \leq N$, and the case $j = N$ and $0 \leq \forall i \leq j$ are trivial. (ii) The case $i = 1$ and $i \leq \forall j \leq N$, i.e. (7), is already proved. (iii) Let us assume (10) holds for $i(< N)$ and $i \leq \forall j \leq N$. We will show (10) holds for $i+1$ and $i+1 \leq \forall j \leq N$. (For $i = N-1$, we have $i+1 = N \leq j = N$. Therefore it is sufficient to consider $i < N-1$ and $j < N$.) Multiply $f^{1,i}(\frac{z_1}{z_3})f^{1,j}(\frac{z_2}{z_3})W^1(z_3)$ from left to (10) with $i \geq 1$ (whose second $f^{a,b}(p^c)W^aW^b$ term in the RHS is replaced by reversed order one $f^{b,a}(p^{-c})W^bW^a$), rewrite $f^{1,j}(\frac{z_2}{z_3})W^1(z_3)W^j(z_2) = f^{j,1}(\frac{z_3}{z_2})W^j(z_2)W^1(z_3) + \cdots$ by using (7), multiply $\frac{1-p}{(1-q)(1-t^{-1})}(1 - p^{-\frac{i+1}{2}}\frac{z_2}{z_1})$, and take a limit $z_3 \to p^{-\frac{i+1}{2}}z_1$. By using (12)–(16) and

(17) (with $j \to j+1$), studying poles carefully and replacing $z_1 \to p^{\frac{1}{2}} z_1$, we obtain (10) with $i \to i+1$ ($2 \leq i+1 \leq j < N$). (iv) Therefore we have proved (10) by induction on i. □

3. Relation to $\mathcal{Z}$-Algebra

Affine Lie algebra $\widehat{\mathfrak{sl}}_N$ is an associative algebra over $\mathbb{C}$ with the Chevalley generators, $e_i^{\pm}$ and h_i ($i = 0, 1, \cdots, N-1$), which satisfy

$$[h_i, h_j] = 0, \quad [h_i, e_j^{\pm}] = \pm a_{ij} e_j^{\pm}, \quad [e_i^{+}, e_j^{-}] = \delta_{ij} h_i, \tag{18}$$

and the Serre relation $(\mathrm{ad}\, e_i^{\pm})^{1-a_{ij}} e_j^{\pm} = 0$ ($i \neq j$), where $(a_{ij})_{0 \leq i,j \leq N-1}$ is the Cartan matrix of $A_{N-1}^{(1)}$ Dynkin diagram.[10] This algebra admits various gradations and we denote its grading operator as d and ρ for the homogeneous and principal gradation respectively, which satisfy

$$\begin{array}{ll} \text{homogeneous gradation} : & [d, e_i^{\pm}] = \pm e_i^{\pm} \delta_{i0}, \\ \text{principal gradation} \quad : & [\rho, e_i^{\pm}] = \pm e_i^{\pm}. \end{array} \tag{19}$$

In current basis $\widehat{\mathfrak{sl}}_N$ is given as follows:

homogeneous gradation

generators : H_n^i, $E_n^{\pm,i}$ ($n \in \mathbb{Z}$, $1 \leq i \leq N-1$), k : center, d : grading operator.

relations :

$$\begin{aligned} &[H_n^i, H_m^j] = k \bar{a}_{ij} n \delta_{n+m,0}, \quad [H_n^i, E_m^{\pm,j}] = \pm \bar{a}_{ij} E_{n+m}^{\pm,j}, \\ &[E_n^{+,i}, E_m^{-,j}] = \delta^{ij} (H_{n+m}^i + kn\delta_{n+m,0}), \quad [d, X_n] = n X_n \ \ (X = H^i, E^{\pm,i}), \end{aligned} \tag{20}$$

and $[E_n^{\pm,i}, E_m^{\pm,j}] = [E_{n-1}^{\pm,i}, E_{m+1}^{\pm,j}]$ and the Serre relations which we omit to write explicitly, where $(\bar{a}_{ij})_{1 \leq i,j \leq N-1}$ is the Cartan matrix of A_{N-1} Dynkin diagram.

principal gradation Let us set $\omega = e^{\frac{2\pi i}{N}}$. Symbol $\equiv$ stands for $\equiv (\mathrm{mod}\ N)$.

generators : β_n ($n \in \mathbb{Z}$, $n \not\equiv 0$), $x_n^{(\mu)}$ ($n \in \mathbb{Z}$, $1 \leq \mu \leq N-1$, μ is understood as $\mathrm{mod}\ N$), k : center, ρ : grading operator.

relations :

$$\begin{aligned} &[\beta_n, \beta_m] = kn\delta_{n+m,0} \quad (n, m \not\equiv 0), \quad [\beta_n, x_m^{(\nu)}] = (1 - \omega^{-\nu n}) x_{n+m}^{(\nu)}, \\ &[x_n^{(\mu)}, x_m^{(\nu)}] = \begin{cases} (\omega^{-\mu m} - \omega^{-\nu n}) x_{n+m}^{(\mu+\nu)} & (\mu + \nu \not\equiv 0) \\ (\omega^{-\mu m} - \omega^{-\nu n}) \beta_{n+m} + kn\omega^{\mu n} \delta_{n+m,0} & (\mu + \nu \equiv 0), \end{cases} \\ &[\rho, X_n] = X_n \ \ (X = \beta, x^{(\mu)}), \end{aligned} \tag{21}$$

and the Serre relations.

Since these two current basis are basis of the same Lie algebra $\widehat{\mathfrak{sl}}_N$, they are related by linear transformation,

$$\beta_{Nm+\nu} = \sum_{i=1}^{N-\nu} E_m^{i,i+\nu} + \sum_{i=N-\nu+1}^{N} E_{m+1}^{i,i+\nu-N},$$

$$x^{(\mu)}_{Nm+\nu} = \sum_{i=1}^{N-\nu} \omega^{\mu(i+\nu-1)} E^{i,i+\nu}_m + \sum_{i=N-\nu+1}^{N} \omega^{\mu(i+\nu-1)} E^{i,i+\nu-N}_{m+1}, \tag{22}$$

$$x^{(\mu)}_{Nm} = \sum_{i=1}^{N-1} \frac{1-\omega^{\mu i}}{1-\omega^{\mu}} H^i_m - \frac{k}{1-\omega^{\mu}} \delta_{m,0},$$

where $m \in \mathbb{Z}$ and $1 \leq \mu, \nu \leq N-1$. Here, for simplicity of the presentation, we have introduced $\widehat{\mathfrak{gl}}_N$ generators $E^{i,j}_n$ $(n \in \mathbb{Z},\ 1 \leq i, j \leq N)$, which satisfy $[E^{i,j}_n, E^{i',j'}_m] = \delta^{ji'} E^{i,j'}_{n+m} - \delta^{ij'} E^{i',j}_{n+m} + \delta^{ij'} \delta^{ji'} kn\delta_{n+m,0}$, and the generators in the homogeneous picture are expressed as $E^{+,i}_n = E^{i,i+1}_n$, $E^{-,i}_n = E^{i+1,i}_n$ and $H^i_n = E^{i,i}_n - E^{i+1,i+1}_n$. We remark $E^{i,j}_n = [E^{i,l}_m, E^{l,j}_{n-m}]$ $(i \neq j)$ and this RHS is independent on l and m.

Next let us consider the splitting of the Cartan part:

$$(\ \widehat{\mathfrak{sl}}_N \text{ generator}) = (\text{exponential of Cartan generators}) \times (\text{new generator}), \tag{23}$$

where new generator commutes with Cartan generators. For homogeneous gradation, the algebra generated by these new generators is known as the ($\mathfrak{sl}_N$ version of) parafermion algebra of level k. For principal gradation, we name it as ($\mathfrak{sl}_N$-version of) $\mathcal{Z}$ algebra of level k, $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$. ($N = 2$ case was studied by Lepowsky and Wilson.[3]) Explicitly the generators of $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$, z^{μ}_n $(n \in \mathbb{Z},\ 1 \leq \mu \leq N-1,\ \mu$ is understood as mod $N)$, are obtained by

$$x^{(\mu)}(\zeta) = :\exp\Big(-\frac{1}{k}\sum_{n\neq 0}\frac{1}{n}(1-\omega^{\mu n})\beta_n \zeta^{-n}\Big): z^{\mu}(\zeta), \tag{24}$$

where $:*:$ stands for the normal ordering for boson β_n and we have introduced currents $x^{(\mu)}(\zeta) = \sum_{n\in\mathbb{Z}} x^{(\mu)}_n \zeta^{-n}$ and $z^{\mu}(\zeta) = \sum_{n\in\mathbb{Z}} z^{\mu}_n \zeta^{-n}$. Then (21) implies the relation of $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$,

$$\begin{aligned} & g^{\mu,\nu}\Big(\frac{\zeta_2}{\zeta_1}\Big) z^{\mu}(\zeta_1) z^{\nu}(\zeta_2) - z^{\nu}(\zeta_2) z^{\mu}(\zeta_1) g^{\nu,\mu}\Big(\frac{\zeta_1}{\zeta_2}\Big) \\ &= \begin{cases} \delta(\omega^{\mu}\frac{\zeta_2}{\zeta_1}) z^{\mu+\nu}(\omega^{\mu}\zeta_2) - \delta(\omega^{-\nu}\frac{\zeta_2}{\zeta_1}) z^{\mu+\nu}(\zeta_2) & (\mu+\nu \not\equiv 0) \\ k\, D\delta(\omega^{\mu}\frac{\zeta_2}{\zeta_1}) & (\mu+\nu \equiv 0)\,, \end{cases} \end{aligned} \tag{25}$$

where $D = \zeta\frac{d}{d\zeta}$, $D\delta(\zeta) = \sum_{n\in\mathbb{Z}} n\zeta^n$ and the structure function $g^{\mu,\nu}(\zeta)$ is given by

$$g^{\mu,\nu}(\zeta) = \exp\Big(-\frac{1}{k}\sum_{\substack{n>0 \\ n\neq 0}}\frac{1}{n}(1-\omega^{\mu n})(1-\omega^{-\nu n})\zeta^n\Big). \tag{26}$$

Next we present an interesting relation between $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ and $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$. Let us consider the following limit[c]:

$$\text{Limit II}: \begin{cases} q = e^{\hbar}, & \hbar \to 0 \\ t = \omega^{-1} q^{\frac{k+N}{N}}, & k : \text{fixed}. \end{cases} \tag{27}$$

[c]For this choice of $t = \omega^{-1}q^{\frac{k+N}{N}}$, we cannot take Limit I because $\beta = \frac{k+N}{N} - \frac{2\pi i}{N\hbar}$ depends on $\hbar$.

We assume that DWA currents $W^i(z)$ have the $\hbar$-expansion

$$W^i(p^{\frac{1-i}{2}}\zeta) = \hbar\omega^{\frac{i}{2}} z^i(\zeta) + O(\hbar^2). \tag{28}$$

Then, under the Limit II, we can show that the relation of $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ (10) reduces to that of $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$ (25). (Eq.(10) begins from $\hbar^2$ term and its coefficient is (25). We remark that in this derivation we do not use free field realization at all.) In other words, $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ with $t = \omega^{-1}q^{\frac{k+N}{N}}$ can be considered as a q-deformation of $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$, which we denote as $\mathrm{DZA}(\widehat{\mathfrak{sl}}_N)_k$,

$$\mathrm{DZA}(\widehat{\mathfrak{sl}}_N)_k = \mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t=\omega^{-1}q^{\frac{k+N}{N}}}. \tag{29}$$

Concerning the free field realization, however, our assumption (28) does not hold on the Fock space except for $N = 2$ case. But calculation of some correlation functions supports the assumption (28); We have checked $\langle\lambda|W^1(\zeta_1)\cdots W^1(\zeta_n)|\lambda\rangle = O(\hbar^n)$ for $n \leq 6$. We guess that the assumption (28) holds on the level of correlation functions, or, on the irreducible representation space obtained by taking some BRST cohomology. For $N = 2$ case, (28) holds on the Fock space, and screening currents and vertex operators of DVA (after some modification of zero mode) reduce to those of ZA.

Finally we mention the character of $\mathrm{DZA}(\widehat{\mathfrak{sl}}_2)_k$ for $k \in \mathbb{Z}_{\geq 2}$, i.e. that of $\mathrm{DVA}_{q,t} = \mathrm{DWA}(\widehat{\mathfrak{sl}}_2)_{q,t}$ with $t = e^{-\pi i}q^{\frac{k+2}{2}}$. We write $W^1(\zeta)$ and $w^1(\lambda)$ as $T(\zeta)$ and λ respectively, e.g., the highest weight state is defined by $T_n|\lambda\rangle = \lambda|\lambda\rangle\delta_{n0}$ $(n \geq 0)$. Since degenerate representations of DVA occur at $\lambda = \lambda_{r,s} = t^{\frac{r}{2}}q^{-\frac{s}{2}} + t^{-\frac{r}{2}}q^{\frac{s}{2}}$,[1] let us consider $\lambda = \lambda_{1,j+\frac{k+2}{2}}$ $(j = -\frac{k}{2}, -\frac{k}{2}+1, \cdots, \frac{k}{2})$ representations. Grading operator ρ satisfies $[\rho, T_n] = nT_n$ and $-\rho|\lambda\rangle = \left(\frac{2j^2+k}{4(k+2)} - \frac{1}{8}\right)|\lambda\rangle$. The character of DZA is defined by $\chi_j^{\mathrm{DZA}}(\tau) = \mathrm{tr}\, y^{-\rho}$ where $y = e^{2\pi i\tau}$ and the trace is taken over irreducible DZA spin j representation space. Shiraishi and present author conjectured[d]

$$\chi_j^{\mathrm{DZA}}(\tau) = y^{\frac{2j^2+k}{4(k+2)}-\frac{1}{8}} \frac{1}{(y;y)_\infty} \sum_{m\in\mathbb{Z}} (-1)^m y^{m(j+\frac{k+2}{2}m)} = y^{\frac{2j^2+k}{4(k+2)}-\frac{1}{8}} \chi^{(2,k+2)}_{1,j+\frac{k+2}{2}}(\tau). \tag{30}$$

Here $\chi^{(p',p'')}_{r,s}(\tau)$ is the Rocha-Caridi character formula[1]

$$\chi^{(p',p'')}_{r,s}(\tau) = \frac{1}{(y;y)_\infty} \sum_{m\in\mathbb{Z}} \left(y^{(p''r-p's+mp'p'')m} - y^{(r+mp')(s+mp'')} \right), \tag{31}$$

which gives the character of the Virasoro minimal representation when p' and p'' are coprime, $(p', p'') = 1$. In the above case, $p' = 2$ and $p'' = k+2$ imply that $(p', p'') = 1$ for odd k but $(p', p'') = 2$ for even k. When q is not a root of unity, by studying the Kac determinant of DVA,[1] we can check that (30) is true. We remark that the character of $\mathrm{DZA}(\widehat{\mathfrak{sl}}_2)_k$, χ_j^{DZA}, coincides with that of $\mathrm{ZA}(\widehat{\mathfrak{sl}}_2)_k$ which is obtained by using the result of Ref. 11, BRST structure of principal $\widehat{\mathfrak{sl}}_2$.

[d]We remark that this character appears in the calculation of the one-point local height probability of the Kashiwara–Miwa model (M. Jimbo, T. Miwa and M. Okado, *Nucl. Phys.* **B275[FS17]** (1986) 517-545).

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Appendix A. DWA$(\widehat{\mathfrak{sl}}_N)_{q,t}$ and ζ-function regularization (by H. Awata, S. Odake and J. Shiraishi)

In this appendix we present an interesting relation between DWA$(\widehat{\mathfrak{sl}}_N)_{q,t}$ and ζ-function regularization.[5]

In string theory,[12] the physical state condition is given by $(L_0-1)|\text{phys}\rangle = 0$ (and its antichiral counterpart), where L_0 is the zero mode of the Virasoro generator. This condition and the space-time dimension are derived by careful study of string theory (Lorentz invariance in the light-cone gauge, nilpotency of BRST charge, etc.), but there is a shortcut method, ζ function regularization method.

First we illustrate this method by taking a bosonic string theory as an example. In the light-cone gauge the Virasoro generator L_n is given by $L_n = \sum_{i=1}^{24}\sum_{m\in\mathbb{Z}}\frac{1}{2} : \alpha^i_{n-m}\alpha^i_m :$ where α^i_n $(n\in\mathbb{Z}, i=1,\cdots,24)$ satisfies $[\alpha^i_n,\alpha^j_m] = n\delta^{ij}\delta_{n+m,0}$ and $: * :$ stands for the normal ordering. The Virasoro zero mode without the normal ordering is $L_0^{\text{noNO}} = \sum_{i=1}^{24}\sum_{n\in\mathbb{Z}}\frac{1}{2}\alpha^i_{-n}\alpha^i_n = \sum_{i=1}^{24}\sum_{n\in\mathbb{Z}}\frac{1}{2} : \alpha^i_{-n}\alpha^i_n : +12\text{“}\sum_{n>0} n\text{”}$. Of course the sum “$\sum_{n>0} n$” is divergent and this expression is meaningless. But we replace the sum “$\sum_{n>0} n$” by $\zeta(-1)$, where $\zeta(z)$ is the Riemann ζ function. Then the above physical state condition is equivalent to the condition that the Virasoro zero mode without the normal ordering annihilates the physical state:

$$L_0^{\text{noNO}}|\text{phys}\rangle = 0, \quad L_0^{\text{noNO}} = L_0 + 12\zeta(-1), \tag{A.1}$$

because of the value $\zeta(-1) = -\frac{1}{12}$. We might say that the Virasoro generator “knows” the value $\zeta(-1)$.

Next let us mimic the above procedure for DWA$(\widehat{\mathfrak{sl}}_N)_{q,t}$ case. DWA current without the normal ordering becomes $W^{i\ \text{noNO}}(z) = \text{“}f^{i,i}(1)^{-\frac{1}{2}}\text{”}\, W^i(z)$, where “$f^{i,i}(1)$” is divergent for generic β (recall $t=q^\beta$ and $q=e^\hbar$). Let a^i_{2m} be coefficients of the following $\hbar$-expansion $(1-q^n)(1-t^{-n})\frac{1-p^{in}}{1-p^n}\frac{1-p^{(N-i)n}}{1-p^{Nn}} = \sum_{m>0} a^i_{2m}(n\hbar)^{2m}$. Then $f^{i,i}(z)$ is $f^{i,i}(z) = \exp\bigl(\sum_{n>0}\frac{1}{n}\sum_{m>0} a^i_{2m}(n\hbar)^{2m}z^n\bigr)$. We define ζ-regularized $f^{i,i}_{\zeta\text{-reg}}(1)$ by exchanging these summations over n and m and replacing $\sum_{n>0} n^{2m-1}$ with $\zeta(1-2m)$ as follows:

$$f^{i,i}_{\zeta\text{-reg}}(1) = \exp\Bigl(\sum_{m>0} a^i_{2m}\zeta(1-2m)\hbar^{2m}\Bigr). \tag{A.2}$$

In the Limit I (4), DWA current behaves as $W^i(z) = \binom{N}{i} + O(\hbar^2)$, which can be

shown by using free field realization.[7] So we require that $\beta = \frac{N+1}{N}$ or $\frac{N}{N+1}$, which corresponds to the vanishing Virasoro central charge, and the zero mode of the i-th DWA current without normal ordering takes the above value $\binom{N}{i}$ on the vacuum state $|\mathrm{vac}\rangle$, which is characterized by $h_n^i|\mathrm{vac}\rangle = 0$ $(n \geq 0, \forall i)$,

$$W_0^{i\ \mathrm{noNO}}|\mathrm{vac}\rangle = \binom{N}{i}|\mathrm{vac}\rangle, \quad W^{i\ \mathrm{noNO}}(z) = f^{i,i}_{\zeta\text{-reg}}(1)^{-\frac{1}{2}}\, W^i(z). \tag{A.3}$$

Since we can show $W_0^i|\mathrm{vac}\rangle = \begin{bmatrix} N \\ i \end{bmatrix}|\mathrm{vac}\rangle$, this requirement implies

$$f^{i,i}_{\zeta\text{-reg}}(1)^{\frac{1}{2}} = \binom{N}{i}^{-1}\begin{bmatrix} N \\ i \end{bmatrix}, \tag{A.4}$$

where $\begin{bmatrix} N \\ i \end{bmatrix} = \frac{[N]!}{[i]![N-i]!}$, $[n]! = [n]\cdots[1]$ and $[n] = \frac{p^{\frac{n}{2}} - p^{-\frac{n}{2}}}{p^{\frac{1}{2}} - p^{-\frac{1}{2}}}$. We can check that this equation really holds by using formulas $\log(\sinh x) = \log x + \sum_{n>0}(-1)^{n-1}\frac{2^{2n-1}B_n}{(2n)!n}x^{2n}$ $(0 < |x| < \pi)$ and $\zeta(1-2m) = (-1)^m\frac{B_m}{2m}$ $(m = 1, 2, \cdots)$. Here B_n is the Bernoulli number defined by $\frac{x}{e^x-1} + \frac{x}{2} = 1 + \sum_{n>0}(-1)^{n-1}\frac{B_n}{(2n)!}x^{2n}$ $(|x| < 2\pi)$. Therefore we might say that $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ (with $t = q^{\frac{N+1}{N}}, q^{\frac{N}{N+1}}$) for each N "knows" all the values $\zeta(1-2m)$ $(m = 1, 2, \cdots)$.

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THE PEIERLS SUBSTITUTION AND THE VANISHING MAGNETIC FIELD LIMIT

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In order to achieve in configuration space a dimensional reduction from dimension two to dimension one, the lowest Landau level (LLL) projection, also called the Peierls substitution, is not sufficient. One has also, once in the LLL, to take the vanishing magnetic field limit.

It is commonly believed that projecting a bidimensional (2d) 1-body system onto the LLL of an external homogeneous magnetic field makes it essentially unidimensional (1d), due to the dimensional reduction of the 1-body phase space from four to two dimensions. Numerous applications have used this line of reasoning, usually referred to as the Peierls substitution.[1] Starting from a 2d Hamiltonian

$$H = -2\partial\bar{\partial} + \omega_c(\bar{z}\bar{\partial} - z\partial) + \frac{1}{2}\omega_c^2 z\bar{z} + V_1(z,\bar{z}) \tag{1}$$

for a particle in a scalar potential $V_1(z,\bar{z})$ coupled to a strong magnetic field (we assume without any loss of generality that $eB \geq 0$, $\omega_c = +eB/2$ is half the cyclotron frequency) and projecting it onto the LLL

$$\psi(z) = f(z)e^{-\frac{\omega_c}{2}z\bar{z}} \tag{2}$$

where $f(z)$ is analytic, one obtains an eigenvalue equation which, in its modern reformulation,[2] rewrites as

$$\left(\omega_c + : V_1(z, \frac{1}{\omega_c}\partial) :\right) f(z) = Ef(z) \tag{3}$$

where the normal ordering :: means that $\frac{1}{\omega_c}\partial$ is put on the left of z. Clearly, the commutative 2d space has been traded for a non commutative space (1d phase space like)

$$[\frac{1}{\omega_c}\partial_z, z] = \frac{1}{\omega_c} \tag{4}$$

However, this system is still bidimensional, as can be readily seen on its partition function, which, in the simplest case $V_1 = 0$, scales like the 2d infinite surface of

the plane. I will argue that in order to achieve an actual dimensional reduction, i.e. not only in phase space but also in configuration space, the LLL projection is not sufficient per se. One has also, once the system has been projected onto the LLL, to take the vanishing magnetic field limit.[3]

Firstly, it might be objected that taking the $B \to 0$ limit in the LLL is counter intuitive: the LLL projection is physically justified when the cyclotron gap is large compared to the temperature and/or the potential ($\hbar\omega_c >> kT, \hbar\omega_c >> V_1$) so that the excited states above the LLL can be ignored. Thus the LLL projection is associated with a strong B limit, and clearly such an interpretation becomes meaningless when the magnetic field vanishes. However, the algorithm proposed here -LLL projection, then $B \to 0$ limit-, which basically amounts to ask about the whereabouts of the LLL Hilbert space in the particular limit when its defining parameter, the magnetic field, vanishes, is well defined mathematically.

Secondly, the $B \to 0$ limit in the LLL might be a priori ambiguous. Still, it can be given a non ambiguous meaning if the system is regularized at long distance, for instance by a harmonic well of frequency ω,[4] and, only after i) projecting onto the LLL harmonic eigenstates -the LLL eigenstates deformed by the harmonic well- ii) letting $B \to 0$, can one take the thermodynamic limit $\omega \to 0$. Under these conditions, we will see that a dimensional reduction of the configuration space from dimension two to one is properly achieved.

Let us first start by a reminder about what is meant by thermodynamic limit in a 2d harmonic well [a]: the 1-body spectrum is

$$E_{nm} = (2n + |m| + 1)\omega \tag{5}$$

with $n \geq 0$, m positive or negative integer. Clearly, the 2d harmonic well partition function

$$Z_\omega = \frac{1}{(2\sinh\frac{\beta\omega}{2})^2} \simeq_{\omega\to 0} \frac{1}{(\beta\omega)^2} \tag{6}$$

has to be identified in the thermodynamic limit, i.e. when $\omega = 0$, with the 2d free partition function $Z_o^{d=2} = V/(2\pi\beta)$, where V is the infinite surface of the 2d plane. Therefore the 2d thermodynamic limit prescription should be that when $\omega \to 0$

$$\frac{1}{(\beta\omega)^2} \simeq \frac{V}{2\pi\beta} \tag{7}$$

Let us now consider, in the presence of a magnetic field, the Landau Hamiltonian here for convenience expressed in the symmetric gauge[b]

$$H_L = -2\partial\bar{\partial} + \omega_c(\bar{z}\bar{\partial} - z\partial) + \frac{1}{2}\omega_c^2 z\bar{z} \tag{8}$$

[a]The first author to use a harmonic well regularization was E. Fermi, see cond-mat 9912229 where the original article of Fermi[5] is translated.

[b]Considering rather the asymmetric gauge where the Landau eigenstates are product of a plane wave on one axis and a Hermite polynomial on the other axis would not help to understand the dimensional reduction mechanism.

and its spectrum

$$E_{nm} = (2n + |m| + 1)\omega_c - m\omega_c \tag{9}$$

with an infinite degeneracy per Landau level $eBV/(2\pi)$ - the product of the infinite surface of the 2d plane by the magnetic field strength. In the LLL, $n = 0, m \geq 0$, and $E = \omega_c$, the 1-body LLL eigenstates are analytic (up to the Landau gaussian factor)

$$(\frac{\omega_c^{m+1}}{\pi m!})^{\frac{1}{2}} z^m e^{-\frac{1}{2}\omega_c z\bar{z}} \tag{10}$$

As already said, the LLL partition function

$$Z_{LLL} = \frac{eBV}{2\pi} e^{-\beta\omega_c} \tag{11}$$

is obviously 2d since it scales like the surface V of the 2d plane.

What happens in the limit $B \to 0$? Here an ambiguity arises due to the vanishing field strength multiplying the infinite surface of the plane. In order to cure this ambiguity, let us confine the system in a harmonic well, so that the 1-body LLL harmonic eigenstates (i.e the deformation of the LLL eigenstates $n = 0, m \geq 0$ by the harmonic well) are still analytic (up to the Landau-harmonic gaussian factor)

$$(\frac{\omega_t^{m+1}}{\pi m!})^{\frac{1}{2}} z^m e^{-\frac{1}{2}\omega_t z\bar{z}} \tag{12}$$

but now the spectrum is non degenerate

$$E = \omega_t + (\omega_t - \omega_c)m \tag{13}$$

with $\omega_t = \sqrt{\omega_c^2 + \omega^2}$. The LLL harmonic partition function becomes

$$Z_{LLL+\omega} = \frac{e^{-\beta\omega_t}}{1 - e^{-\beta(\omega_t - \omega_c)}} \tag{14}$$

At this point, one should keep ω fixed, let $B \to 0$, and then take the thermodynamic limit $\omega \to 0$.

Before doing so, let us check that by keeping B fixed but taking the thermodynamic limit $\omega \to 0$, one correctly recovers Z_{LLL}. Since, when $\omega \to 0$, $\omega_t - \omega_c \simeq \omega^2/(2\omega_c)$ one indeed gets, using the 2d thermodynamic limit prescription (7),

$$Z_{LLL+\omega} \underset{\omega \to 0}{\simeq} \frac{e^{-\beta\omega_c}}{\beta(\omega_t - \omega_c)} \to_{\omega=0} \frac{eB}{2\pi} V e^{-\beta\omega_c} = Z_{LLL} \tag{15}$$

Note that what we found here is yet an other way to actually show that the Landau degeneracy is, in the thermodynamic limit, $eBV/(2\pi)$.

Now in the case of interest, first set $B = 0$, i.e. $\omega_t = \omega$, then take the thermodynamic limit $\omega \to 0$, one gets, still using (7),

$$Z_{LLL+\omega} = \frac{e^{-\frac{\beta\omega}{2}}}{2\sinh\frac{\beta\omega}{2}} \underset{\omega \to 0}{\simeq} \frac{1}{\beta\omega} \to_{\omega=0} \sqrt{\frac{V}{2\pi\beta}} = Z_o^{d=1} \tag{16}$$

i.e. the 1d partition function for a free particle on a line of infinite length $L = \sqrt{V}$. At the level of the spectrum, being in the LLL and a harmonic well, and taking, as advocated above, the $B \to 0$ limit, the LLL harmonic basis (12) and spectrum (13) have narrowed down to

$$(\frac{\omega^{m+1}}{\pi m!})^{\frac{1}{2}} z^m, \quad m \geq 0 \tag{17}$$

and

$$\omega(m+1), \quad m \geq 0 \tag{18}$$

This amounts to pick up on each 2d harmonic energy level $(j+1)\omega$, $j \geq 0$, with degeneracy $j+1$, the eigenstate of maximal angular momentum j, and consequently zero radial quantum number, yielding the spectrum (18) which happens to coincide with a 1d harmonic spectrum, provided that the 2d positive angular momentum quantum number m is now interpreted as the 1d harmonic quantum number.

It is therefore manifest, both on the partition function and on the spectrum, that a dimensional reduction from d=2 to d=1 has been achieved. To put it bluntly, we have shown that

$$Z_{LLL} = \frac{eB}{2\pi} V e^{-\beta\omega_c} \underset{B\to 0}{\to} Z_o^{d=1} = \sqrt{\frac{V}{2\pi\beta}} \tag{19}$$

and accordingly for the density of states

$$\rho_{LLL}(E) = \frac{eB}{2\pi} V \delta(E - \omega_c) \underset{B\to 0}{\to} \rho_o^{d=1}(E) = \frac{\sqrt{V}}{\pi\sqrt{2E}} \tag{20}$$

where $\rho_{LLL}(E)$ and $\rho_o^{d=1}(E)$ stand respectively for the LLL and the free 1d density of states. Clearly, setting directly $B = 0$ in Z_{LLL} or in $\rho_{LLL}(E)$ has no meaning whatsoever. Still, (19,20) have been given a non ambiguous meaning through the long distance harmonic regularization.

Up to now one has dealt with spectra and partition functions. The same logic applies as well to the Hamiltonian and the eigenstates: consider (1) but now in a harmonic well

$$H = -2\partial\bar{\partial} + \omega_c(\bar{z}\bar{\partial} - z\partial) + \frac{1}{2}\omega_t^2 z\bar{z} + V_1(z, \bar{z}) \tag{21}$$

and project it onto the LLL-harmonic basis

$$\psi(z) = f(z) e^{-\frac{\omega_t}{2} z\bar{z}} \tag{22}$$

to obtain the eigenvalue equation[6]

$$\left(\omega_t + (\omega_t - \omega_c) z\partial + : V_1(z, \frac{1}{\omega_t}\partial) :\right) f(z) = E f(z) \tag{23}$$

When $B \to 0$ it becomes

$$\omega \left(1 + z\partial + : V_1(z, \frac{1}{\omega}\partial) :\right) f(z) = E f(z) \tag{24}$$

acting on $\psi(z) = f(z)e^{-\frac{\omega}{2}z\bar{z}}$. It is obvious that the kinetic part of the Hamiltonian (24) is nothing but the 1d harmonic well Hamiltonian in a coherent state representation, with the mapping $z^m \to H_m(x)$ between the 2d analytic function z^m and the 1d Hermite polynomial $H_m(x)$. Moreover, looking at the potential V_1, one realizes that the 2d commuting space has been traded for a non commutative space

$$[\frac{1}{\omega}\partial_z, z] = \frac{1}{\omega} \tag{25}$$

In the thermodynamic limit, $\omega \to 0$, one gets an infinite non commutativity which should be viewed as the signature of the dimensional reduction which has taken place from a 2d to a 1d system.

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APPLICATIONS OF A HARD-CORE BOSE–HUBBARD MODEL TO WELL-DEFORMED NUCLEI*

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An exactly solvable hard-core Bose–Hubbard model, which is equivalent to a mean-filed plus nearest-level pairing theory, for a description of well-deformed nuclei is used and applied to the actinide region. Binding energies and pairing excitation energies of $^{226-234}$Th, $^{230-240}$U, and $^{236-243}$Pu isotopes are calculated and compared with the corresponding experimental values.

1. Introduction

Pairing is an important residual interaction in nuclear physics. Typically, after adopting a mean-field approach, the pairing interaction is treated approximately using either Bardeen–Cooper–Schrieffer (BCS) or Hartree–Fock–Bogolyubov (HFB) methods, sometimes in conjunction with correction terms evaluated within the Random-Phase Approximation (RPA). However, both BCS and HFB approximations suffer from serious difficulties, the nonconservation of the number of particles being one that can lead to serious problems, such as spurious states, nonorthogonal solutions, etc. Another problem with these approximations is related to the fact that both BCS and the HFB methods break down for an important class of physical situations. A remedy in terms of particle number projection complicates the algorithms considerably, often without yielding a better description of higher-lying excited states that are a natural part of the spectrum of the pairing Hamiltonian. Over the past few years progress has been made in the development of better algorithms that bypass the Bogolyubov transformation and thus are free of problems related to particle number nonconservation.[1,2] In these approximation, either a configuration-energy truncation scheme or a many-body Fock-space basis cutoff was used, so the results were still not exact.

*Dedicated to Professor Wu F. Y. on his 70th Birthday Celebration

Exact solutions of the mean-filed plus pairing model were first studied for the equal strength pairing model.[3–5] Recently, generalizations that include state dependent pairing have been considered.[6–9] In these cases, the Bethe ansatz was used, from which excitation energies and the corresponding wavefunctions can be determined through a set of nonlinear equations. Unfortunately, solving these nonlinear equations is not practical when the number of levels and valence nucleon pairs are large, which is usually the case for well-deformed nuclei.

2. A Hard-core Bose-Hubbard Model for nuclei

In Ref. 9, a hard-core Bose-Hubbard model was proposed, which is equivalent to a mean-field plus nearest-level pairing theory. As is well known, an equal strength pairing interaction, which is used in many applications, is not a particularly good approximation for well-deformed nuclei. In Ref. 2, a level-dependent Gaussian-type pairing interaction with

$$G_{ij} = Ae^{-B(\varepsilon_i - \varepsilon_j)^2} \tag{1}$$

was used, where i and j each represent doubly occupied levels with single-particle energies ϵ_i and ϵ_j. The parameters $A < 0$ and $B > 0$ are adjusted in such a way that the location of the first excited eigen-solution lies approximately at the same energy as for the constant pairing case. Of course, there is some freedom in adjusting the parameters, allowing one to control in a phenomenological way the interaction among the levels. Expression (1) implies that scattering between particle pairs occupying levels with single-particle energies that lie close are favored; scattering between particle pairs in levels with distant single-particle energies are unfavored. As an approximation, this pairing interaction was further simplified to nearest-level coupling in Ref. 9, namely, G_{ij} is given by (1) if the levels i and j lie adjacent to one another in energy, with G_{ij} taken to be 0 otherwise. Hence, the Hamiltonian can be expressed as

$$\hat{H} = \sum_i \varepsilon_i + \sum_{i,j}{}' t_{ij} b_i{}^+ b_j, \tag{2}$$

where the first sum runs over the orbits occupied by a single fermion which occurs in the description of odd-A nuclei or broken pair cases, and the second primed sum runs only over levels that are occupied by pairs of fermions. For the nearest-level pairing interaction case the t-matrix is given by $t_{ii} = 2\epsilon_i + G_{ii} = 2\epsilon_i + A$ and $t_{ii+1} = t_{i+1i} = G_{ii+1}$ with $t_{ij} = 0$ otherwise. The fermion pair operators in this expression are given by

$$b_i{}^+ = a_i{}^+ a_{\bar{i}}{}^+, \quad b_i = a_{\bar{i}} a_i, \tag{3}$$

where ${a_i}^+$ is the i-th level single-fermion creation operator and $a_{\bar{i}}^+$ the corresponding time-reversed state. The ${b_i}^+$ and b_i satisfy the following commutation relation:

$$[b_i, {b_j}^+] = \delta_{ij}(1 - 2N_i), \quad [N_i, {b_j}^+] = \delta_{ij}{b_j}^+, \quad [N_i, b_j] = -\delta_{ij}b_j \tag{4}$$

where $N_i = \frac{1}{2}({a_i}^+ a_i + {a_{\bar{i}}}^+ a_{\bar{i}})$ is the pair number operator in the i-th level for even-even nuclei.

In this paper the Nilsson Hamiltonian is used to generate the mean-field. In this case there is at most one valence nucleon pair or a single valence nucleon in each level due to the Pauli principle. Equivalently, these pairs can be treated as bosons with projection onto the subspace with no doubly occupied levels.[9]

The eigenstates of (2) for k-pair excitation can be expressed as

$$|k; \xi, (n_{j_1}, n_{j_2}, \cdots, n_{j_r}) n_f\rangle = {\sum}'_{i_1 < i_2 \cdots < i_k} C^{(\xi)}_{i_1 i_2 \cdots i_k} \times$$

$$ {b_{i_1}}^\dagger {b_{i_2}}^\dagger \cdots {b_{i_k}}^\dagger |(n_{j_1}, n_{j_2}, \cdots, n_{j_r}) n_f\rangle, \tag{5}$$

where $j_1, j_2, \cdots, j_r$ are the levels occupied by r single particles, the prime indicates that $i_1, i_2, \cdots, i_k$ can not be taken to be $j_1, j_2, \cdots, j_r$ in the summation, and n_f is the total numbers of single valence nucleons, that is $n_f = \sum_j n_j$. Since only even-even and odd-A nuclei are treated without including broken pair cases in this paper, r is taken to be 1 for odd-A nuclei, and 0 for even-even nuclei. In Eq. (5), $C^{(\xi)}_{i_1 i_2 \cdots i_k}$ is a determinant given by

$$\begin{vmatrix} g^{\xi_1}_{i_1} & g^{\xi_1}_{i_2} & \cdots & g^{\xi_1}_{i_k} \\ g^{\xi_2}_{i_1} & g^{\xi_2}_{i_2} & \cdots & g^{\xi_2}_{i_k} \\ \cdots & \cdots & \cdots & \cdots \\ g^{\xi_k}_{i_1} & g^{\xi_k}_{i_2} & \cdots & g^{\xi_k}_{i_k} \end{vmatrix}, \tag{6}$$

where ξ is a shorthand notation for a selected set of k eigenvalues of the t matrix without the corresponding r rows and columns denoted as $\tilde{t}$, which can be used to distinguish the eigenstates with the same number of pairs, k, and g^{ξ_p} is the p-th eigenvector of the $\tilde{t}$ matrix.

The excitation energies corresponding to (5) can be expressed as

$$E^{(\xi)}_k = \sum_{i=1}^{r} \varepsilon_{j_i} + \sum_{j=1}^{k} E^{(\xi_j)}, \tag{7}$$

where the first sum runs over r Nilsson levels each occupied by a single valence nucleon, which occurs in odd-A nuclei or in broken pair cases, the second one is a

sum of k different eigenvalues of the $\tilde{t}$-matrix. Obviously, $\tilde{t}$ is a $(k-r)\times(k-r)$ matrix, since those orbits occupied by single valence nucleons are excluded resulting from the Pauli blocking. $E^{(\xi_p)}$ is the p-th eigenvalue of the $\tilde{t}$-matrix, that is

$$\sum_j \tilde{t}_{ij} g_j{}^{\xi_p} = E^{(\xi_p)} g_i{}^{\xi_p}. \tag{8}$$

Hence

$$\hat{H}|k;\xi,(n_{j_1},n_{j_2},\cdots,n_{j_r})n_f\rangle = \sum_{i_1<i_2<\cdots<i_k}\sum_{\mu=1}^{k}\sum_{P}(-)^P\Big(\sum_{i=1}^{r}\varepsilon_{j_i}+E^{(\xi_{P(\mu)})}\Big)\times$$

$$g_{i_1}^{(\xi_{P(1)})}g_{i_2}^{(\xi_{P(2)})}\cdots g_{i_\mu}^{(\xi_{P(\mu)})}\cdots g_{i_k}^{(\xi_{P(k)})}b_{i_1}{}^{\dagger}b_{i_2}{}^{\dagger}\cdots b_{i_k}{}^{\dagger}|(n_{j_1},n_{j_2},\cdots,n_{j_r})n_f\rangle$$

$$= E_k^{(\xi)}|k;\xi,(n_{j_1},n_{j_2}\cdots n_{j_k})n_f\rangle, \tag{9}$$

where P runs over all permutations, $E^{(\xi_\mu)}$ is the μ-th eigenvalue of the $\tilde{t}$ matrix. Eq. (9) is valid for any k. If one assumes that the total number of orbits is N for even-even nuclei, the k-pair excitation energies are determined by the sum of k different eigenvalues chosen from the N eigenvalues of the $\tilde{t}$ matrix with $r=0$, the total number of excited levels is $N!/k!(N-k)!$. While for odd-A nuclei or broken pair cases, the levels that are occupied by the single valence nucleons should be excluded in the original t matrix. In the latter case, the eigenvalue problem (4) can be solved simply by diagonalizing the corresponding $\tilde{t}$ matrix as shown in Eq. (9).

3. Applications to Actinide Isotopes

In this section, we try to describe nuclei in the actinide region with the mean-field plus nearest-level pairing model using the axial-symmetric Nilsson potential as the mean-field. Other than what is manifest through the mean field, the quadrupole-quadrupole interaction is not considered. In this case, exact solutions can be obtained by using the above simple method. As for the binding energy, the contributions from real quadrupole-quadrupole interaction is expected to be relatively small.[10] This conclusion applies to low-lying 0^+ excited states as well as ground states. As shown in Ref. 11, contributions from the pairing interaction is very important to the low-lying excited 0^+ states in these deformed regions. Hence, the position of low-lying 0^+ states is an estimate based on the Nilsson mean field plus pairing approximation.

In this well-deformed region there are a lot of nuclei. The parameters were fixed by considering the $^{226-234}$Th, $^{230-240}$U, and $^{236-243}$Pu isotopes. Specifically, the binding energies of these isotopes were calculated. Table 1 shows the binding energy results as well as pairing excitation energies of the theory for $^{226-234}$Th, $^{230-240}$U,

and $^{236-243}$Pu, with the corresponding experimental values taken from Ref. 12. The parameters A and B in Eq. (1) were fit as follows to maximize agreement with experiment:

$$A = \alpha_1 + \beta_1 k + \gamma_1 n_f, \quad B = \alpha_2 + \beta_2 k + \gamma_2 n_f, \tag{10}$$

where α_i, β_i, and γ_i are parameters that were fit for each isotope.

Table 1. Calculated binding and pairing excitation energies are compared with the corresponding experimental values for various $^{226-234}$Th, $^{230-240}$U, and $^{236-243}$Pu isotopes. $B_{\rm th}$(MeV) and $B_{\rm exp}$(MeV) denote, respectively, the theoretical and experimental binding energies.[12]

Nucleus	Spin and Parity	B_{exp}(MeV)	B_{th}(MeV)	Pairing excitation Energies of Exp. (MeV)		Pairing excitation Energies of Th. (MeV)	
^{226}Th	0^+	-1730.54	-1732.17	0_2^+	0.805	0_2^+	0.999
				$\frac{1}{2}^+_2$	3.226	$\frac{1}{2}^+_2$	1.299
^{227}Th	$\frac{1}{2}^+$	-1736.00	-1733.97	$\frac{1}{2}^+_3$	5.188	$\frac{1}{2}^+_3$	1.391
				$\frac{1}{2}^+_4$	6.495	$\frac{1}{2}^+_4$	1.415
^{228}Th	0^+	-1743.10	-1739.30	0_2^+	0.831	0_2^+	0.718
				$\frac{5}{2}^+_2$	0.029	$\frac{5}{2}^+_2$	0.057
^{229}Th	$\frac{5}{2}^+$	-1748.36	-1744.42	$\frac{5}{2}^+_3$	0.317	$\frac{5}{2}^+_3$	0.516
^{230}Th	0^+	-1755.16	-1756.90	0_2^+	0.635	0_2^+	1.199
				$\frac{5}{2}^+_2$	0.241	$\frac{5}{2}^+_2$	0.907
^{231}Th	$\frac{5}{2}^+$	-1760.27	-1764.21	$\frac{5}{2}^+_3$	0.302	$\frac{5}{2}^+_3$	1.204
				$\frac{5}{2}^+_4$	0.317	$\frac{5}{2}^+_4$	1.230
				0_2^+	0.730	0_2^+	1.647
^{232}Th	0^+	-1766.71	-1768.66	0_3^+	1.079	0_3^+	2.585
^{233}Th	$\frac{1}{2}^+$	-1771.50	-1772.92	$\frac{1}{2}^+_2$	0.310	$\frac{1}{2}^+_2$	0.907
				0_2^+	0.810	0_2^+	1.066
^{234}Th	0^+	-1777.69	-1779.81	0_3^+	1.150	0_3^+	2.562
				0_4^+	1.470	0_3^+	2.904
^{231}U	$\frac{5}{2}^-$	-1758.72	-1761.26		—	$\frac{5}{2}^-_2$	0.646
^{232}U	0^+	-1760.00	-1758.94	0_2^+	0.691	0_2^+	0.961
				$\frac{5}{2}^+_2$	0.340	$\frac{5}{2}^+_2$	0.732
^{233}U	$\frac{5}{2}^+$	-1771.74	-1770.23	$\frac{5}{2}^+_3$	0.546	$\frac{5}{2}^+_3$	0.803
				0_2^+	0.809	0_2^+	0.747
^{234}U	0^+	-1778.59	-1774.41	0_3^+	1.044	0_3^+	0.933
				0_4^+	1.781	0_4^+	1.696
				$\frac{7}{2}^-_2$	0.670	$\frac{7}{2}^-_2$	0.826
^{235}U	$\frac{7}{2}^-$	-1783.89	-1780.23	$\frac{7}{2}^-_3$	0.700	$\frac{7}{2}^-_3$	1.056

Table 1 (Continued)

Nucleus	Spin and Parity	B_{exp}(MeV)	B_{th}(MeV)	Pairing excitation Energies of Exp. (MeV)		Pairing excitation Energies of Th. (MeV)	
				0_2^+	0.919	0_2^+	0.913
^{236}U	0^+	-1790.44	-1786.71	0_3^+	2.155	0_3^+	1.186
				0_4^+	2.750	0_4^+	2.319
				2_2^+	0.846	2_2^+	0.586
^{237}U	$\frac{1}{2}^+$	-1795.56	-1795.48	2_3^+	0.905	2_3^+	0.700
				0_2^+	0.925	0_2^+	0.877
^{238}U	0^+	-1801.715	-1802.22	0_3^+	0.993	0_3^+	2.874
				2_2^+	0.193	2_2^+	0.185
^{239}U	$\frac{5}{2}^+$	-1806.52	-1810.23	2_3^+	0.734	2_3^+	0.459
				2_4^+	0.757	2_4^+	0.786
^{240}U	0^+	-1812.45	-1815.41	—		0_2^+	0.100
^{236}Pu	0^+	-1790.46	-1792.36	0_2^+	3.000	0_2^+	0.645
				2_2^-	0.691	2_2^-	0.617
^{237}Pu	$\frac{7}{2}^-$	-1795.56	-1795.87	2_3^-	0.696	2_3^-	2.173
				0_2^+	0.942	0_2^+	0.407
				0_3^+	1.134	0_3^+	1.987
^{238}Pu	0^+	-1801.72	-1799.96	0_4^+	1.229	0_4^+	2.170
				0_5^+	1.427	0_5^+	2.681
^{239}Pu	$\frac{1}{2}^+$	-1806.52	-1805.12	2_2^+	0.753	2_2^+	0.354
				0_2^+	0.860	0_2^+	1.030
^{240}Pu	0^+	-1812.45	-1810.68	0_3^+	1.089	0_3^+	2.144
				0_4^+	1.526	0_4^+	2.626
				2_2^+	0.233	2_2^+	0.088
^{241}Pu	$\frac{5}{2}^+$	-1816.64	-1816.09	2_3^+	0.801	2_3^+	0.587
^{242}Pu	0^+	-1822.41	-1821.89	0_2^+	0.956	0_2^+	1.186
				2_2^-	0.333	2_2^-	0.845
^{243}Pu	$\frac{7}{2}^-$	-1826.63	-1828.63	2_3^-	0.450	2_3^-	1.146
				2_4^-	0.742	2_4^-	1.815

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ELLIPTIC ALGEBRA AND INTEGRABLE MODELS FOR SOLITONS ON NONCOMMUTATIVE TORUS $\mathcal{T}$

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We study the algebra $\mathcal{A}_n$, the basis of the Hilbert space $\mathcal{H}_n$ in terms of θ functions of the positions of n solitons. Then we embed the Heisenberg group as the quantum operator factors in the representation of the transfer matrices of various integrable models. Finally we generalize our result to the generic θ case.

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1. Solitons on noncommutative plane

In the noncommutative plane R^2, the coordinates x^1 and x^2 satisfy the following relation:

$$[x^i, x^j] = i\theta, \tag{1}$$

here θ is a constant. The algebra $\mathcal{A}$ associated with this space is generated by the functions of x^1 and x^2. The functional form of the algebra $\mathcal{A}$ is defined by the Moyal $*$ product

$$f * g(x) = e^{i\epsilon_{ij}\theta \frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j}} f(x)g(y)|_{x=y}. \tag{2}$$

The derivative ∂_i is the infinitesimal translation automorphism of the algebra $\mathcal{A}$:

$$x^i \longrightarrow x^i + \varepsilon^i, \tag{3}$$

where ε^i is a c-number. For algebra $\mathcal{A}$ this automorphism is internal:

$$\partial_i f(x) = i\theta\epsilon_{ij}[x^j *, f(x)] = i\theta_{ij}[x^j *, f(x)], \tag{4}$$

here $\theta_{ij} = \theta\epsilon_{ij}$

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The operator form of $\mathcal{A}$ is generated by Weyl–Moyal transformation.

$$a^\dagger = \frac{1}{\sqrt{2\theta}}(x^1 + ix^2), \qquad a = \frac{1}{\sqrt{2\theta}}(x^1 - ix^2), \tag{5}$$

which obey

$$[a, a^\dagger] = 1. \tag{6}$$

Since a and $a^\dagger$ satisfy the commutation relations of the creation and annihilation operators, we can identify the function $f(x^1, x^2)$ as the functions of a and $a^\dagger$ acting on the standard Fock space $\mathcal{H}$ of the creation and annihilation operators:

$$\mathcal{H} = \{|0\rangle, |1\rangle, \cdots, |n\rangle, \cdots\}. \tag{7}$$

where $|0\rangle$ and $|n\rangle$ satisfy:

$$a|0\rangle = 0, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad a^\dagger a|n\rangle = n|n\rangle. \tag{8}$$

The Weyl–Moyal transformation maps the ordinary commutative functions onto operators in the Fock space $\mathcal{H}$:

$$f(x) = f(z = x^1 - ix^2, \bar{z} = x^1 + ix^2) \longrightarrow$$
$$\hat{f}(a, a^\dagger) = \int \frac{d^2x d^2y}{(2\pi)^2} f(x) e^{i[\bar{p}(\sqrt{2\theta}a - z) + p(\sqrt{2\theta}a^\dagger - \bar{z})]}, \tag{9}$$

where:

$$p = \frac{y^1 + iy^2}{2}, \quad \bar{p} = \frac{y^1 - iy^2}{2}. \tag{10}$$

It is easy to see that if

$$f \longrightarrow \hat{f}, \quad g \longrightarrow \hat{g}, \tag{11}$$

then

$$f * g \longrightarrow \hat{f}\hat{g} \tag{12}$$

and

$$\int d^2x f(x) \longrightarrow \pi\theta Tr\hat{f}(a, a^\dagger). \tag{13}$$

The translations of R^2 are generated by $\hat{\partial}_i$ which are isomorphism to $\mathcal{A}$ while applying on the Fock space $\mathcal{H}$:

$$\hat{\partial}_i \longleftrightarrow i\theta_{ij}\hat{x}_j. \tag{14}$$

In paper 4, Harvey, Kraus and Larsen introduced a quasi-unitary operator to generate various soliton solutions in noncommutative geometry. In noncommutative plane R^2, this operator is defined as

$$T \equiv \frac{a^\dagger}{\sqrt{a^\dagger a}}. \tag{15}$$

Acting this operator T on the basis of the Hilbert space $\mathcal{H}$, we have

$$T|n\rangle = |n+1\rangle, \quad \langle n|T^\dagger = \langle n+1|. \tag{16}$$

and

$$T|n\rangle\langle n|T^\dagger = |n+1\rangle\langle n+1|. \tag{17}$$

This means that

$$TP_nT^\dagger = P_{n+1}, \tag{18}$$

where $P_n = |n\rangle\langle n|$ denote the projection operator onto the n-th states and $P_n^2 = P$. Thus we have

$$TT^\dagger|n\rangle = |n\rangle, (n \leq 1) \quad \text{and} \quad TT^\dagger|0\rangle = 0, \tag{19}$$

and

$$TT^\dagger = 1 - |0\rangle\langle 0| = 1 - P_0. \tag{20}$$

T is the quasi-unitary soliton generating operator.

2. Solitons on noncommutative torus $\mathcal{T}$ and Heisenberg group

In the noncommutative torus $\mathcal{T}$, the algebra $\mathcal{A}$ is generated by the Wilson loop $\hat{U}_i$, $(i = 1, 2)$. The arbitrary element $a \in \mathcal{A}$ is

$$a = \sum_{j_1 j_2} c_{j_1,j_2} U_1^{j_1} U_2^{j_2} \tag{21}$$

For the periodicities l and $2\pi l\tau$ of the torus, the generators of the algebra $\mathcal{A}$ are

$$U_1 = e^{ilx^2}, \quad U_2 = e^{il(\tau_2 x^1 - \tau_1 x^2)}. \tag{22}$$

Since $[x^1, x^2] = i\theta$ locally, so

$$\hat{U}_1\hat{U}_2 = \hat{U}_2\hat{U}_1 e^{il^2\tau_2\theta}. \tag{23}$$

Now let us consider the integral torus case $\frac{l^2\tau_2\theta}{2\pi} = A \in \mathbf{N}$ (or $\mathbf{Z}_+$) i. e. the normalized area A of the torus is an integer. Then the Wilson loop U_1 and U_2 are commutative

$$U_1U_2 = U_2U_1. \tag{24}$$

We orbifold $\mathcal{T}$ into $\frac{\mathcal{T}}{n\times n} = \mathcal{T}_n$ by introducing

$$W_i = (U_i)^{\frac{1}{n}}, \tag{25}$$

then on $\mathcal{H}_n$, the Hilbert space on $\mathcal{T}_n$, we will have noncommutative algebra $\mathcal{A}_n$ generated by

$$W_1W_2 = W_2W_1 e^{\frac{2\pi i}{n}} \equiv W_2W_1\omega \tag{26}$$

which satisfy

$$W_1^n = W_2^n = 1 \tag{27}$$

where $\omega = e^{\frac{2\pi i}{n}}$.

The Basis vectors of the Hilbert space $\mathcal{H}_n$ are

$$V_a = \sum_{b=1}^{n} F_{-a,b}, (a = 1, 2, \cdots, n),$$

$$F_\alpha \equiv F_{\alpha_1,\alpha_2} = e^{i\pi n\alpha_2} \prod_{j=1}^{n} \sigma_{\alpha_1,\alpha_2}(z_j - \frac{1}{n}\sum_{k=1}^{n} z_k), \tag{28}$$

here $\alpha \equiv (\alpha_1, \alpha_2) \in Z_n \times Z_n$, and

$$\sigma_\alpha(z) = \theta \begin{bmatrix} \frac{1}{2} + \frac{\alpha_1}{n} \\ \frac{1}{2} + \frac{\alpha_2}{n} \end{bmatrix} (z, \tau). \tag{29}$$

The θ function can be transformed to a operator form by the Weyl–Moyal transformation:

$$\theta(z) = \sum_m e^{i\pi m^2\tau + 2\pi i m z} \to \theta(\hat{z}) = \sum_m e^{i\pi m^2\tau} : U_1^m U_2^m : \tag{30}$$

Since

$$W_i : U_1^m U_2^m := \omega^{\pm m} : U_1^m U_2^m : \tag{31}$$

we have

$$W_1 V_a(z_1, \cdots, z_n) = \left(\prod_{i=1}^{n-1} T_{\frac{\tau}{n}}^{(i)} \right) T_{\frac{\tau}{n}-\tau}^{(n)} V_a(z_1, \cdots, z_n) \tag{32}$$

$$W_2 V_a(z_1, \cdots, z_n) = \left(\prod_{i=1}^{n-1} T_{\frac{1}{n}}^{(i)} \right) T_{\frac{1}{n}-1}^{(n)} V_a(z_1, \cdots, z_n), \tag{33}$$

where

$$T_a^{(i)} f(z) = f(z_1, \cdots, z_i + a, \cdots, z_n). \tag{34}$$

Substituting the expressions of V_a we get

$$\begin{aligned} W_1 V_a(z_1, \cdots, z_n) &= V_{a+1}(z_1, \cdots, z_n), \\ W_2 V_a(z_1, \cdots, z_n) &= e^{2\pi i \frac{a}{n}} V_a(z_1, \cdots, z_n). \end{aligned} \tag{35}$$

Then the algebra

$$\mathcal{A}_n = \{W^\alpha \equiv W^{\alpha_1\alpha_2} = W_1^{\alpha_1} W_2^{\alpha_2}\} \tag{36}$$

is realized as the $2^n \times 2^n$ Heisenberg matrices I^α,

$$(I_\alpha)_{ab} = \delta_{\alpha+a_1,a_2} \omega^{ba_2} \tag{37}$$

Corresponding to the ∂_i on R^2, we have a $su_n(\mathcal{T}_n)$ acting on $\mathcal{H}_n$[18]

$$su_n(\mathcal{T}_n) : \{E_\alpha | \alpha \neq (0, 0)\}. \tag{38}$$

Here

$$E_\alpha = (-1)^{\alpha_1}\sigma_\alpha(0)\sum_j \prod_{k\neq j}\frac{\sigma_\alpha(z_{jk})}{\sigma_0(z_{jk})}\left[\frac{l}{n}\sum_{i\neq j}\frac{\sigma'_\alpha(z_{ji})}{\sigma_\alpha(z_{ji})} - \partial_j\right],$$
$$\alpha \equiv (\alpha_1, \alpha_2) \neq (0,0) \equiv (n,n), \tag{39}$$

and

$$E_0 = -\sum_j \partial_j, \tag{40}$$

where $z_{jk} = z_j - z_k$, $\partial_j = \frac{\partial}{\partial z_j}$. The commutation relation between E_α and E_γ is

$$[E_\alpha, E_\gamma] = (\omega^{-\alpha_2\gamma_1} - \omega^{-\alpha_1\gamma_2})E_{\alpha+\gamma}, \tag{41}$$

or in more common basis, let $E_{ij} \equiv \sum_{\alpha\neq 0}(I^\alpha)_{ij}E_\alpha$, we have

$$[E_{jk}, E_{lm}] = E_{jm}\delta_{kl} - E_{lk}\delta_{jm}. \tag{42}$$

This commutation rule can also be obtained from the quasiclassical limit of the representation of the Sklyanin algebra.[19]

Since the Wilson loops W_1 and W_2 acting on the noncommutative covering torus $\mathcal{T}$ is to shift z_i to $(z_i + \frac{\tau}{n} - \delta_{in}\tau)$ and $(z_i + \frac{1}{n} - \delta_{in})$ respectively, we can get the automorphism of $E_\beta \in su_2(\mathbf{T})$ by noncommutative gauge transformation $w^\alpha \in \mathcal{A}$

$$W_1 E_\alpha(z_i)W_1^{-1} = \omega^{-\alpha_2}E_\alpha(z_i), \tag{43}$$
$$W_2 E_\alpha(z_i)W_2^{-1} = \omega^{\alpha_1}E_\alpha(z_i). \tag{44}$$

Let $E_\alpha \in g$ to act on V_a, we find that

$$E_\alpha V_a = \sum_b (I_\alpha)_{b}aV_a. \tag{45}$$

Next, we know that

$$W_\alpha V_a = \sum_b (I_\alpha)_{ba}V_b, \tag{46}$$

so on $\mathcal{H}_n$, we establish the isomorphism:

$$su_n(\mathbf{T}) \Longleftrightarrow \mathcal{A}; \quad E_\alpha \longleftrightarrow W_\alpha. \tag{47}$$

The operator form of the projection operators becomes

$$\frac{1}{n}\sum_\beta W^{0\beta}(I_\beta)_{ii} = P_i = |V_i\rangle\langle V_i| \tag{48}$$

and the ABS operators is simply

$$E_{10} \cong W_1 = \sum_a |V_{a+1}\rangle\langle V_a| \tag{49}$$

3. The integrable models for the solitons on noncommutative torus $\mathcal{T}$

In this section, we will embed the $su_n(\mathcal{T})$ derivative operators as the "quantum" operator factors in the representation of the transfer matrix (Lax operator) of the various integrable models i.e.

The elliptic Gaudin model on noncommutative space[20] is defined by the transfer matrix (quantum Lax operator):

$$L^G_{ij}(u) = \sum_{\alpha\neq(0,0)} w_\alpha(u) E_\alpha(I_\alpha)_{ij} \tag{50}$$

where $w_\alpha(u) = \frac{\theta'(0)\sigma_\alpha(u)}{\sigma_\alpha(0)}$ and E_α and I_α are the generators of $su(n)$ (or $\mathcal{A}_{n-1}$ Weyl) and $G_{\mathcal{H}}(n)$ respectively. This transfer matrix can also be obtained as the nonrelativistic limit of the Ruijsenaars–Macdonald operators. The common eigenfunctions and eigenvalues of Gaudin model is solved in terms of the Bethe ansatz.[21] Now we substitute the difference representation of $su(n)$ E_α (39) into (50), we get a factorized L of the Gaudin model

$$\begin{aligned} L_G(u)^i_j &= E_0 + \sum_{\alpha\neq(0,0)} E_\alpha(I_\alpha)^i_j \\ &= \sum_k \phi(u,z)^i_k \phi^{-1}(u,z)^k_j \partial_u - l\sum_k \partial_u\phi(u,z)^i_k \phi^{-1}(u,z)^k_j, \end{aligned} \tag{51}$$

where the factors are the vertex face intertwiner

$$\phi(u,z)^i_j = \theta\begin{bmatrix} \frac{1}{2} - \frac{i}{n} \\ \frac{1}{2} \end{bmatrix}(u + nz_j - \sum_k z_k + \frac{n-1}{2}, n\tau). \tag{52}$$

For the Gaudin model on noncommutative torus, the z_i is the origin (position) of the i-th soliton, ∂_i as its infinitesimal translation is equivalently to $[z_i*,]$.

Next, the elliptic Calogero–Moser model is defined by the Hamiltonian:

$$H = \sum_{i=1}^n \partial_i^2 + \sum_{i\neq j} g\wp(z;j) \tag{53}$$

where $\wp(z) = \partial^2\sigma(z)$. The corresponding Lax operator is

$$L_{CM}(u)^i_j = (p_i - \frac{l}{n}\frac{\partial}{\partial q_i}\ln\Delta(z))\delta^i_j - \frac{l}{n}\sigma'(0)(1-\delta^i_j)\frac{\sigma(u+z_{ji})}{\sigma(u)\sigma(z_{ji})} \tag{54}$$

This Lax operator can be gauge transformed into the factorized L (51) of the Gaudin model by the following matrix:

$$G(u;z)^i_j \equiv \frac{\phi(u;z)^i_j}{\prod_{l\neq j}\theta_{\frac{1}{2},\frac{1}{2}}(z_{jl})} \tag{55}$$

The C.M. model gives the dynamics of a long distance interaction between n-bodies located at z_i $(i = 1,\cdots,n)$. On noncommutative torus, it gives the dynamics of n solitons and z_i becomes the position of the center of the i-th soliton. According

to Ref. 9, the interaction between n-solitons is the Laplacian of a Kähler potential K, which is the logarithm of a Vandermonde determinant. Actually we have

$$\sum_{i\neq j}\wp(z;j)=\sum_i\partial_i^2\log\prod_{j\neq k}\sigma(z_j-z_k)\equiv\sum_i\partial_i^2K(u,z) \tag{56}$$

and

$$e^{K(u,z)}=\prod_{j\neq k}\sigma(z_j-z_k)\sigma(nu+\frac{n-1}{n})=\det(\phi_k^j)\equiv\sigma(nu+\frac{n-1}{2})\prod_{i\neq j}\sigma(z_i-z_j). \tag{57}$$

The variable u of the marked torus is the spectral parameter or evaluation parameter of Lax matrix K_j^i.

This Ruijsenaars operators are related to the quantum Dunkle operators and the q-deformed Kniznik–Zamolodchikov–Bernard equations. The eigenfunctions could be also expressed in terms of double Bloch wave as the algebraic geometric methods.[22] We will show this in the more familiar formalism of the elliptic quantum group.

4. The $Z_n \times Z_n$ Heisenberg group in case of the general θ

For the generic θ case, as in paper 23 we find that $\theta\tau=\eta$, here η is the crossing parameter and the $Z_n\times Z_n$ Heisenberg group of shift of solitons is realized by the Sklyanin algebra $\mathcal{S}_{\tau,\eta}$. The noncommutative algebra $\mathcal{A}$ is realized as Elliptic quantum group $E_{\tau,\eta}$. The evaluation module of $E_{\tau,\eta}$ is expressed by the Boltzmann weight of the IRF model.

$$R(u,\lambda)=\sum_{i=1}^n E_{i,i}\otimes E_{i,i}+\sum_{i\neq j}\alpha(u,\lambda_{ij})E_{i,i}\otimes E_{j,j}+\sum_{i\neq j}\beta(u,\lambda_{ij})E_{i,j}\otimes E_{j,i} \tag{58}$$

where

$$\alpha(u,\lambda)=\frac{\theta(u)\theta(\lambda+\eta)}{\theta(u-\eta)\theta(\lambda)},\qquad \beta(u,\lambda)=\frac{\theta(u+\lambda)\theta(\eta)}{\theta(u-\eta)\theta(\lambda)} \tag{59}$$

It satisfies the dynamical YBE:

$$\begin{aligned}&R(u_1,u_2,\lambda-\eta h^{(3)})^{12}R(u_1,\lambda)^{13}R(u_2,\lambda-\eta h^{(1)})^{23}\\&=R(u_2,\lambda)^{23}R(u_1,\lambda-\eta h^{(2)})^{13}R(u_1-u_2,\lambda)^{12}\end{aligned} \tag{60}$$

where $R(u,\lambda-\eta h^{(3)})^{12}$ acts on a tensor $v_1\otimes v_2\otimes v_3$ as $R(u,\lambda-\eta\mu)\otimes Id$ if v_3 has weight μ.

The elliptic quantum group $E_{\tau,\eta}(sl_n)$ is an algebra generated by a meromorphic function of a variable h and a matrix $L(z,\lambda)$ with noncommutative entries:

$$\begin{aligned}&R(u_1-u_2,\lambda-\eta h^{(3)})^{12}L(u_1,\lambda)^{13}L(u_2,\lambda-\eta h^{(1)})^{23}\\&=L(u_2,\lambda)^{23}L(u_1,\lambda-\eta h^{(2)})^{13}R(u_1-u_2,\lambda)^{12}.\end{aligned} \tag{61}$$

here $L(z,\lambda)$ gives an evaluation representation of the quantum group

$$L(u,\lambda)_k^j = \frac{\sigma_0(u+\frac{\xi}{n}-\eta\delta-\eta a_{kj}-\frac{n-1}{2})}{\sigma_0(u-\eta\delta-\frac{n-1}{2})}\prod_{i\neq j}\frac{\sigma_0(-\frac{\xi}{n}+\eta a_{ki})}{\sigma_0(\eta a_{ji})} \tag{62}$$

The Transfer matrix of IRF is expressed by the Ruijsenaars operators which gives the dynamics of solitons

$$T(u)f(\lambda) = \sum_{i=1}^{N} L_{ii}(u,\lambda)f(\lambda-\eta h) \tag{63}$$

and the Ruijsenaars–Macdonald operator M is

$$M = \sum_{i}^{N}\prod_{j:j\neq i}\frac{\theta(\lambda_i-\lambda_j+l\eta)}{\theta(\lambda_i-\lambda_j)}T_i \tag{64}$$

So we have

$$T_i f(\lambda) = f(\lambda_i-\eta b) \tag{65}$$

Then the Hilbert space of non-commutative torus becomes the common eigenvectors of the transfer matrix.

The wave functions have the form

$$\psi = \prod_i e^{c_i z_i}\prod\theta(z_i+t_i-\eta) \tag{66}$$

which will be twisted by η when z_i changed by Wilson loop U_1, U_2.

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NEW RESULTS FOR SUSCEPTIBILITIES IN PLANAR ISING MODELS*

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We briefly review recent progress on calculating susceptibilities in planar Ising models.

1. Introduction

First of all, a project like the current one cannot be undertaken by a single person. We owe a lot to our collaborators, teachers, and colleagues, especially R.J. Baxter, H.W. Capel, A.J. Guttmann, M. Jimbo, B.-Q. Jin, X.-P. Kong, T. Miwa, B.M. McCoy, B.G. Nickel, W.P. Orrick, M. Sato, and T.T. Wu. The literature on the two-dimensional Ising model also is very extended. Therefore, we shall only give limited citations, and encourage the interested reader to consult the quotations in these references. Most of the current work is a brief review of results in Refs. 1–4.

The symmetric two-dimensional Ising model is defined by

$$\mathcal{H} = -J \sum_{m,n} (\sigma_{m,n}\sigma_{m,n+1} + \sigma_{m,n}\sigma_{m+1,n}). \tag{1}$$

For this model it is convenient to define elliptic modulus[5]

$$k = 1/\sinh^2(2J/k_{\mathrm{B}}T), \tag{2}$$

which is < 1 for $T < T_{\mathrm{c}}$ and > 1 for $T > T_{\mathrm{c}}$, with $k \to 1/k$ giving the Kramers-Wannier duality transformation.

The spontaneous magnetization is simply given by[6,7]

$$\langle\sigma\rangle = \begin{cases} (1-k^2)^{1/8}, & T < T_{\mathrm{c}}, \\ 0, & T \geq T_{\mathrm{c}}. \end{cases} \tag{3}$$

The calculation of the pair correlation function

$$C(m,n) = \langle\sigma_{0,0}\sigma_{m,n}\rangle \tag{4}$$

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is more involved and can be carried out using quadratic difference equations[8]

$$\begin{aligned}&\big[C(m,n+1)C(m,n-1)-C(m,n)^2\big]\\&\quad+k\left[C^*(m+1,n)C^*(m-1,n)-C^*(m,n)^2\right]=0,\end{aligned}\tag{5}$$

$$\begin{aligned}&\big[C(m+1,n)C(m-1,n)-C(m,n)^2\big]\\&\quad+k\left[C^*(m,n+1)C^*(m,n-1)-C^*(m,n)^2\right]=0,\end{aligned}\tag{6}$$

where $C^*(m,n)$ is the dual correlation function obtained by replacing $k\to 1/k$. For the symmetric case (1), these two equations are equivalent. To solve them we need initial conditions. For $T=T_{\rm c}$, we have

$$C(n,n)=C^*(n,n)=\prod_{j=1}^{n}\frac{\Gamma(j)^2}{\Gamma(j+\frac{1}{2})\Gamma(j-\frac{1}{2})}.\tag{7}$$

which form was already known to Onsager and Kaufman.[9] For $T\neq T_{\rm c}$, $C(n,n)$ and $C^*(n,n)$ can be calculated by Toeplitz determinants[1,4,7]

$$C(n,n)=(-1)^n\det_{1\le i,j\le n}\big(\{a_{i-j-1}\}\big),\tag{8}$$

$$C^*(n,n)=\det_{1\le i,j\le n}\big(\{a_{i-j}\}\big),\tag{9}$$

where

$$a_n=(2nk^{-1}a_{n-1}+a_{-n})/(2n+1),\tag{10}$$

$$a_{-n-1}=(2nka_{-n}+a_{n-1})/(2n+1),\tag{11}$$

for $n=1,2,\cdots$, with the initial conditions

$$a_0=\frac{2}{\pi k}\left[\mathrm{E}(k)-(1-k^2)\mathrm{K}(k)\right],\quad a_{-1}=-\frac{2}{\pi}\,\mathrm{E}(k).\tag{12}$$

However, it can be done faster by another set of quadratic difference equations due to Jimbo and Miwa.[10]

2. High- and Low-Temperature Series for Susceptibility

Very recently, with the help of (5) the high- and low-temperature series for the susceptibility were much extended by the authors of Ref. 2. In terms of the reduced susceptibility,

$$\bar\chi\equiv k_{\rm B}T\chi=\sum_{m,n=-\infty}^{\infty}\big(\langle\sigma_{0,0}\sigma_{m,n}\rangle-\langle\sigma_{0,0}\rangle^2\big),\tag{13}$$

they found for $T>T_{\rm c}$,

$$\begin{aligned}\bar\chi=\;&1+4s_{\rm h}+12s_{\rm h}^2+32s_{\rm h}^3+76s_{\rm h}^4+176s_{\rm h}^5+400s_{\rm h}^6+\cdots\\&+2007330258829172991431166572284107035662323251845 3\backslash\\&\quad 6754555022644572376340673830115916010858599831857 6\,s_{\rm h}^{323},\end{aligned}\tag{14}$$

with $s_h \equiv s/2 \equiv \sinh(2K)/2$, and for $T < T_c$,

$$\begin{aligned}\bar{\chi} = \; & 4s_l^4 + 16s_l^6 + 104s_l^8 + 416s_l^{10} + 2224s_l^{12} + \cdots \\ & + 3051547724509044350855662072500389468463893273907\backslash \\ & \quad 5732810211229434299420849612234517174982030845245\backslash \\ & \quad 5331887458424846630637797467206682914215700492366\backslash \\ & \quad 9271259707379855275224873707435550114462001144064\, s_l^{646}, \end{aligned} \tag{15}$$

with $s_l \equiv 2/s \equiv 2/\sinh(2K)$. The size of the coefficients may look ridiculous at first sight. However, it is well-known to series expanders that the new information in each successive coefficient is often in the last few digits.

Near the ferromagnetic critical point, the susceptibility behaves asymptotically as

$$\frac{\beta^{-1}\chi_{\pm}}{(\sqrt{1+\tau^2}+\tau)^{1/2}} \approx C_{0\pm}(2K_c\sqrt{2})^{7/4}|\tau|^{-7/4}\hat{F}_{\pm} + \hat{B}_f, \tag{16}$$

where $(\sqrt{1+\tau^2}+\tau)^{1/2} = 1/\sqrt{s}$ and $\tau = (1/s - s)/2$, and $\pm$ stands for T above or below T_c. In (16) the ferromagnetic background is given by[2]

$$\begin{aligned}\hat{B}_f = \; & (-0.10413324509383102645216012686047343371623672731 4 \\ & \quad -0.07436886975320708001995859169799500328047632028\tau \\ & \quad -0.00814471390911959953715428586557238932660577 40\tau^2 \\ & \quad +0.00450410771223201592635502085298697059136452 8\tau^3 \\ & \quad +\cdots - 0.16279253648974618861881216566686\tau^{14}) \\ & +(\log|\tau|)\times \\ & \quad (0.03235226847730940609065652672122166663773094889 8\tau \\ & \quad -0.00577552937968846300914875640132010136771529 80\tau^3 \\ & \quad +\cdots - 0.04142858646305286935680314413762 0\tau^{14}) \\ & +(\log|\tau|)^2\times \\ & \quad (0.00939156987114587213179533187270757706495136 54\tau^4 \\ & \quad -0.00869592546287923802156416645191752987912922\tau^6 \\ & \quad +\cdots - 0.00555710021511613080348969643146 79\tau^{14}) \\ & +(\log|\tau|)^3\times \\ & \quad (-0.00001577156913845184048000101262146173817 8\tau^9 \\ & \quad +0.00003442820662088875536477998568577533 80\tau^{11} \\ & \quad -0.00005244271774872261741615837791493 93\tau^{13}), \end{aligned} \tag{17}$$

whereas, the ferromagnetic scaling amplitudes functions are given by[2]

$$\begin{aligned}\hat{F}_+ \; & = 1 + \tau^2/2 - \tau^4/12 - 0.1235292285752086663\tau^6 \\ & \quad +0.136610949809095\tau^8 - 0.13043897213\tau^{10} + \cdots, \text{ for } T > T_c, \end{aligned} \tag{18}$$

$$\begin{aligned}\hat{F}_- \; & = 1 + \tau^2/2 - \tau^4/12 - 6.3213068404959366230 67\tau^6 \\ & \quad +6.25199747046024329\tau^8 - 5.6896599756180\tau^{10} + \cdots, \text{ for } T < T_c. \end{aligned} \tag{19}$$

More coefficients are given in Ref. 2. The last digit in each term above may not be reliable. As we have normalized[a] $F_{\pm}(\tau) \to 1$ for $T \to T_c$ (or $\tau \to 0$), we need to give also the leading susceptibility amplitudes:

$$\begin{aligned} C_0^+ &= 1.00081526044021264711947636304721\backslash \\ &\qquad 0\,236937534925597789\,(2K_c\sqrt{2})^{-7/4}\sqrt{2}, \\ C_0^- &= 1.00096032872526218948093495517209\backslash \\ &\qquad 7\,320572505951770117\,(2K_c\sqrt{2})^{-7/4}\sqrt{2}/(12\pi). \end{aligned} \tag{20}$$

Near the antiferromagnetic critical point, the susceptibility behaves as

$$\frac{\beta^{-1}\chi}{\left(\sqrt{1+\tau^2}+\tau\right)^{1/2}} \approx \hat{B}_{\rm af}, \tag{21}$$

where

$$\begin{aligned} \hat{B}_{\rm af} = {} & 0.15886652296094748823335923136902101169252390084 16 \\ & +0.14956683693853590519438202943359128637471120726 2\tau \\ & +0.01071222587983288033470968550659996768542030678\tau^2 \\ & +\cdots+0.00712367768251120814903247637966 7\tau^{14} \\ & +(\log|\tau|) \\ & \quad(-0.15531719015801105859341335389327345299921216003 05\tau \\ & \quad +0.03206714814586975221843437287457551882247161782\tau^3 \\ & \quad +\cdots-0.00940562303807656077194749250886 49\tau^{14}) \\ & +(\log|\tau|)^2 \\ & \quad(0.01153371437882328027949011442761203640684043805\tau^4 \\ & \quad -0.01131173492069156006753505653220784271640568 4\tau^6 \\ & \quad +\cdots-0.00674470189451526288478200059343432\tau^{14}) \\ & +(\log|\tau|)^3 \\ & \quad(0.00005789971947648772977600672211440622495 41\tau^9 \\ & \quad -0.00016991508824012890240796446744935908812\tau^{11} \\ & \quad +0.00032664884687465587957270016883093909\tau^{13}). \end{aligned} \tag{22}$$

The difference of $\hat{F}_+$ and $\hat{F}_-$ in Eqs. (18) and (19) implies that a suggestion of Aharony and Fisher[11] breaks down in higher order. They had brought up the possibility that there are "no irrelevant variables." This they concluded from the speculation that the Ising model free energy in the critical region can be described entirely by two nonlinear but "analytic" (thermal and magnetic) scaling fields. Then the scaling amplitude can be found to be

$$\hat{F}_{\pm} = 1 + \frac{1}{2}\tau^2 - \frac{31}{384}\tau^4 + \frac{125}{3072}\tau^6 + \mathrm{O}(\tau^8), \tag{23}$$

equal above and below T_c.[2]

[a]Note that we have a slight change of notation with respect to Ref. 2, as we have rescaled all B's and F's with a factor $\sqrt{s}$.

We now know that this simple picture is incomplete and that corrections to scaling due to breaking of rotational symmetry must be considered. Indeed, the correlation functions have a kind of multipole long-distance expansion,[12] which can explain the deviations from fourth order on. Very recently, a conformal field theory explanation has also been given.[13] To study the effect in more detail we shall have to study the model on other lattices.

Another interesting feature discussed in Ref. 2 is that the susceptibility has a natural boundary at the critical point, i.e. there exists a closed curve of (essential) singularities fully prohibiting analytic continuation in the complex temperature plane from high to low temperatures. The Ising susceptibility is not differentiably finite, unlike the zero-field free energy and the spontaneous magnetization. This then explains why there is no simple closed form expression available after half a century of research. Yet, we now have algorithms of polynomial complexity, which is as good for numerical analysis.

3. Baxter's Z-invariant inhomogeneous Ising model

Baxter's Z-invariant Ising model is defined in terms of a set of oriented straight lines carrying "rapidity" variables u_i, v_j, $\cdots$. In the scaling limit the scaled correlation function depends on a single distance variable R, as first discovered by Bai-Qi Jin,[1]

$$R = \frac{1}{2}\left[\left\{\sum_{j=1}^{2m}\cos(2u_j)\right\}^2 + \left\{\sum_{j=1}^{2m}\sin(2u_j)\right\}^2\right]^{1/2}. \tag{24}$$

This is given in terms of the $2m$ rapidity variables crossing between the two spins in question. Using the diagonal correlation length $\xi_{\rm d}$ to introduce the scaled distance

$$r = R/\xi_{\rm d}, \quad \text{where} \quad \xi_{\rm d}^{-1} = |\log k|, \tag{25}$$

we have found the most general form of the scaled correlation functions to be

$$\langle\sigma\sigma'\rangle \approx |1-k^{-2}|^{1/4}F(r), \quad \langle\sigma\sigma'\rangle^* \approx |1-k^{-2}|^{1/4}G(r), \tag{26}$$

where the functions $F(r)$ and $G(r)$ satisfy

$$FF'' - F'^2 = -r^{-1}GG', \qquad GG'' - G'^2 = -r^{-1}FF', \tag{27}$$

and the front factor is the square of the spontaneous magnetization for $T < T_{\rm c}$ or $k > 1$. $F(r)$ and $G(r)$ are the Painlevé functions for the uniform rectangular Ising lattice,[14] see Refs. 1, 4 for more details.

4. Susceptibility in Z-Invariant Lattice

For a general ferromagnetic Z-invariant lattice with $\mathcal{N}$ sites, the susceptibility χ is given by

$$\bar{\chi} \equiv k_{\rm B}T\chi = \lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\sum_{m_1,n_1}\sum_{m_2,n_2}\left(\langle\sigma_{m_1,n_1}\sigma_{m_2,n_2}\rangle - \langle\sigma_{0,0}\rangle^2\right), \tag{28}$$

where (m_1, n_1) and (m_2, n_2) run through the possible coordinates of the spins. In periodic cases one of the two sums can be done trivially. In quasiperiodic cases this can only be done asymptotically at the largest distance scale. Hence, in the scaling limit and for both periodic and quasiperiodic Z-invariant lattices, $\bar{\chi}$ becomes

$$\bar{\chi} \approx g_0 \int_{-\infty}^{+\infty} \mathrm{d}M \int_{-\infty}^{+\infty} \mathrm{d}N \, \frac{\check{F}_{\pm}(\kappa R)}{R^{1/4}}, \tag{29}$$

where

$$\frac{\check{F}_{-}(\kappa R)}{R^{1/4}} = |1-k^{-2}|^{1/4}\big(G(R/\xi_{\mathrm{d}}) - 1\big), \tag{30}$$

$$\frac{\check{F}_{+}(\kappa R)}{R^{1/4}} = |1-k^{-2}|^{1/4}\, F(R/\xi_{\mathrm{d}}), \tag{31}$$

$\kappa = 1/\xi_{\mathrm{d}} = |\log k|$, and R reduces to

$$R = \sqrt{aM^2 + 2bMN + cN^2} \tag{32}$$

with a, b, and c known constants that can be calculated choosing suitable integer coordinates M and N. Also, g_0 is the corresponding multiplicity factor counting how many spin distance vectors fall exactly or asymptotically within a unit cell in the (M, N) plane. Therefore,[4]

$$\bar{\chi} = \frac{2\pi g_0}{\sqrt{ac-b^2}} \int_0^{\infty} \mathrm{d}r\, r^{3/4} \check{F}_{\pm}(r)\, \kappa^{-7/4} + \cdots = A_{\pm}|t|^{-7/4} + \mathrm{O}(|t|^{-3/4}), \tag{33}$$

with $t \equiv |T - T_{\mathrm{c}}|/T_{\mathrm{c}}$, giving the exact $T > T_{\mathrm{c}}$ and $T < T_{\mathrm{c}}$ susceptibility amplitudes for all periodic and quasiperiodic Z-invariant lattices.

Note that this result implies that the ratio A_+/A_- is universal for all periodic and quasiperiodic ferromagnetic Z-invariant Ising models. This may be the first time that this is shown to this generality for the magnetic susceptibility. For the analysis of the long susceptibility series in the isotropic square lattice A_+ and A_- were evaluated to very high precision by Nickel.

Therefore, we can now give A_+ and A_- for the isotropic square (sq), triangular (tr) and honeycomb (hc) lattices to many places, i.e.[4]

$$\begin{aligned}
A_+^{\mathrm{sq}} &= 0.962581732308772114044329809433469495167139194757936 5,\\
A_+^{\mathrm{tr}} &= 0.924206958245164329697157577855931717669626152002838 9,\\
A_+^{\mathrm{hc}} &= 1.046417076152338359733871672674357433252295746539088,\\
A_-^{\mathrm{sq}} &= 0.025536974522023905385953456226398471929219687270774 55,\\
A_-^{\mathrm{tr}} &= 0.024518904477000004890808552397197720236530228514229 50,\\
A_-^{\mathrm{hc}} &= 0.027761098425397045077433797952582855036099698776332 51.
\end{aligned} \tag{34}$$

Also, more generally,

$$\bar{\chi} = \frac{2\pi g_0}{\sqrt{ac-b^2}}\, |\log(k)|^{-7/4} \int_0^{\infty} \mathrm{d}r\, r^{3/4} \check{F}_{\pm}(r) \tag{35}$$

is a product of a factor depending on rapidities and the modulus and a factor which is a universal integral over a Painlevé V function. Hence, the amplitudes

are known—in principle—to this high accuracy for all Z-invariant (quasi)-periodic cases. We plan to use these values later to analyze long series for the isotropic triangular and honeycomb lattices, once they are available.

We note that the numbers given above agree to a few places with earlier series extrapolations. Four of the six agree to about ten places with those of Wu et al.[14] and of Vaidya.[15] For T above T_c, they agree to better than three places with those obtained from the Syozi-Naya[16] approximation, but this can be understood as this approximation is precisely the $\chi_<^{(1)}$ approximation in Wu et al.

5. Outlook

We are working to extend and analyze series for other lattices in order to get more information on irrelevant variables in the corrections to scaling, having a preliminary algorithm of polynomial complexity for the isotropic honeycomb and triangular lattices which reproduces the known series coefficients. But more work needs to be done to increase its efficiency, as we will need to go to one to two hundred terms, before being able to see clearly the effect of the irrelevant variables.

We are also looking at the susceptibility of Ising models on Penrose tilings. Finally, we also want to look at the effect of frustration, which occurs in the regime where elliptic modulus k is purely imaginary.

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ALGEBRAIC GEOMETRY AND HOFSTADTER TYPE MODEL

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In this report, we study the algebraic geometry aspect of Hofstadter type models through the algebraic Bethe equation. In the diagonalization problem of certain Hofstadter type Hamiltonians, the Bethe equation is constructed by using the Baxter vectors on a high genus spectral curve. When the spectral variables lie on rational curves, we obtain the complete and explicit solutions of the polynomial Bethe equation; the relation with the Bethe ansatz of polynomial roots is discussed. Certain algebraic geometry properties of Bethe equation on the high genus algebraic curves are discussed in cooperation with the consideration of the physical model.

1. Introduction

It is known for the past decade that algebraic geometry has played a certain intriguing role in certain 2-dimensional solvable statistical lattice models, a notable example would be the chiral Potts N-state integrable model (see e.g., Refs. 1,3 and references therein). In the note, we report the algebraic geometry aspect of another model of physical interest in solid state physics. In the early 90's, motivated by the work of Wiegmann and Zabrodin[12] on the appearance of $U_q(sl_2)$ symmetry in problems of magnetic translation, Faddeev and Kashaev[6] pursued the diagonalization problem on the following Hamiltonian by the quantum transfer matrix method which was developed by the Leningrad school in the early eighties:

$$H_{FK} = \mu(\alpha U + \alpha^{-1}U^{-1}) + \nu(\beta V + \beta^{-1}V^{-1}) + \rho(\gamma W + \gamma^{-1}W^{-1}) \ , \qquad (1)$$

where U, V, W are unitary operators with the Weyl commutation relation for a primitive N-th root of unity ω and the N-th power identity property, $UV = \omega VU$, $VW = \omega WV$, $WU = \omega UW$; $U^N = V^N = W^N = 1$. As a special limit case for $\rho = 0$, the model is reduced to the (rational flux) Hofstadter Hamiltonian, a model

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possessing several physical interpretations with the history which can trace back to the work of Peierls[11] on Bloch electrons in metals with the presence of a constant external magnetic field. By the pioneering works of the 50s and 60s,[2,5,7,9,13] the role of magnetic translations was found, and it began a systematic study of this 2D lattice model. In 1976, Hofstadter[8] found the butterfly figure of the spectral band versus the magnetic flux which exhibits a beautiful fractal picture. Here the phase of ω represents the magnetic flux (per plaquette). In Ref. 6, a general frame work to determine the eigenvalues of certain quantum chains appeared in the transfer matrix was presented. The method relies on a special monodromy solution of the Yang-Baxter equation for the six-vertex R-matrix; this solution appeared also in the study of chiral Potts model.[3] For a finite size L, the trace of the monodromy matrix gives rise to the transfer matrix acting on the quantum space $\overset{L}{\otimes}\mathbf{C}^N$; while the Hofstadter type Hamiltonian (1) can be realized in the case $L = 3$. In general, the diagonalization problem of the transfer matrix can be formulated into the Bethe equation through the Baxter vector[a], visualized on a "spectral" curve associated to the corresponding model. In Ref.,[10] we presented a detailed and rigorous mathematical study on the Bethe equation associated to the Hofstadter type model. In particular, we obtained the complete solution of the Bethe equation for models with rational spectral curves for $L \leq 3$, among which a special Hofstadter type of H_{FK} in Ref. 6 is included, and further expended to all the other sectors. In this note, we explain the main results we have obtained in Ref. 10; detailed derivations, as well as extended references to the literature, may be found in that work.

This paper is organized as follows. In Sect. 2, we first recall results in transfer matrix relevant to our discussion; then introduce the Bethe equation (or Baxter T-Q equation) through the Baxter vector on the spectral curve. In Sect. 3, we consider the case when the spectral data lie on rational curves and perform the mathematical derivation of the answer. We present the complete solutions of the Bethe polynomial equations of all sectors for $L \leq 3$. In Sect. 4, we discuss the "degeneracy" relation between the Bethe solutions and the eigenspaces in the quantum space of the transfer matrix for $L = 3$; also its connection with the usual Bethe ansatz technique in literature, in particular the result obtained in Ref. 6. In Sect. 5, we describe the algebraic geometry properties of the high genus spectral curve arisen from the Hofstadter Hamiltonian.

Notations. The letters $\mathbf{Z}, \mathbf{R}, \mathbf{C}$ will denote the ring of integers, real, complex numbers respectively, $\mathbf{N} = \mathbf{Z}_{>0}$, $\mathbf{Z}_N = \mathbf{Z}/N\mathbf{Z}$. Throughout this report, N will always denote an odd positive integer with $M = [\frac{N}{2}]$: $N = 2M + 1, M \geq 1$; ω is a primitive N-th root of unity, and $q := \omega^{\frac{1}{2}}$ with $q^N = 1$, i.e., $q = \omega^{M+1}$. An element v in the vector space $\mathbf{C}^N$ is represented by a sequence of coordinates, $v_k, k \in \mathbf{Z}$, with the N-periodic condition, $v_k = v_{k+N}$, i.e., $v = (v_k)_{k\in\mathbf{Z}_N}$. The standard basis of $\mathbf{C}^N$ will be denoted by $|k\rangle$, with the dual basis of $\mathbf{C}^{N*}$ by $\langle k|$ for $k \in \mathbf{Z}_N$. For a positive

[a]It is also called as the "Baxter vacuum state" in other literature.

integers n, we denote $\overset{n}{\otimes}\ \mathbf{C}^N$ the tensor product of n-copies of the vector space $\mathbf{C}^N$. We use the notation of ρ-shifted factorials: $(a;\rho)_n = (1-a)(1-a\rho)\cdots(1-a\rho^{n-1})$ for $n \in \mathbf{N}$, and $(a;\rho)_0 = 1$.

2. Transfer Matrix and the Bethe Equation

We consider the Weyl algebra generated by the operators Z, X satisfying the Weyl commutation relation with the N-th power identity, $ZX = \omega XZ, Z^N = X^N = I$, and denote $Y := ZX$. In the canonical irreducible representation of the Weyl algebra, the operators Z, X, Y act on $\mathbf{C}^N$ with the expressions: $Z(v)_k = \omega^k v_k$, $X(v)_k = v_{k-1}$, $Y(v)_k = \omega^k v_{k-1}$. It is known that the following L-operator for an element $h = [a : b : c : d]$ of the projective 3-space $\mathbf{P}^3$ with operator-entries acting on the quantum space $\mathbf{C}^N$,

$$L_h(x) = \begin{pmatrix} aY & xbX \\ xcZ & d \end{pmatrix} , \quad x \in \mathbf{C} ,$$

possesses the intertwining property of the Yang-Baxter relation,

$$R(x/x')(L_h(x) \bigotimes_{aux} 1)(1 \bigotimes_{aux} L_h(x')) = (1 \bigotimes_{aux} L_h(x'))(L_h(x) \bigotimes_{aux} 1)R(x/x') , \quad (2)$$

where $R(x)$ is the matrix of a 2-tensor of the auxiliary space $\mathbf{C}^2$ with the following numerical expression,

$$R(x) = \begin{pmatrix} x\omega - x^{-1} & 0 & 0 & 0 \\ 0 & \omega(x - x^{-1}) & \omega - 1 & 0 \\ 0 & \omega - 1 & x - x^{-1} & 0 \\ 0 & 0 & 0 & x\omega - x^{-1} \end{pmatrix} .$$

By performing the matrix product on auxiliary spaces and the tensor product of quantum spaces, one has the L-operator associated to an element $\vec{h} = (h_0, \ldots, h_{L-1}) \in (\mathbf{P}^3)^L$, $L_{\vec{h}}(x) = \bigotimes_{j=0}^{L-1} L_{h_j}(x)$, which again satisfies the relation (2). The entries of $L_{\vec{h}}(x)$ are operators of the quantum space $\overset{L}{\otimes}\ \mathbf{C}^N$, and its trace defines the commuting transfer matrices for $x \in \mathbf{C}$, $T_{\vec{h}}(x) = \mathrm{tr}_{aux}(L_{\vec{h}}(x))$. The transfer matrix $T_{\vec{h}}(x)$ can also be computed by changing L_{h_j} to $\widetilde{L}_{h_j}$ via a gauge transformation: $\widetilde{L}_{h_j}(x, \xi_j, \xi_{j+1}) = A_j L_{h_j}(x) A_{j+1}^{-1}, 0 \le j \le L-1$, with $A_j = \begin{pmatrix} 1 & \xi_j - 1 \\ 1 & \xi_j \end{pmatrix}$ and $A_L := A_0$. One has

$$\widetilde{L}_{h_j}(x, \xi_j, \xi_{j+1}) = \begin{pmatrix} F_{h_j}(x, \xi_j - 1, \xi_{j+1}) & -F_{h_j}(x, \xi_j - 1, \xi_{j+1} - 1) \\ F_{h_j}(x, \xi_j, \xi_{j+1}) & -F_{h_j}(x, \xi_j, \xi_{j+1} - 1) \end{pmatrix} ,$$

where $F_h(x, \xi, \xi') := \xi' aY - xbX + \xi'\xi xcZ - \xi d$. Hence $T_{\vec{h}}(x) = \mathrm{tr}_{aux}(\widetilde{L}_{\vec{h}}(x, \vec{\xi}))$, $\vec{\xi} := (\xi_0, \ldots, \xi_{L-1})$ where

$$\widetilde{L}_{\vec{h}}(x, \vec{\xi}) := \bigotimes_{j=0}^{L-1} \widetilde{L}_{h_j}(x, \xi_j, \xi_{j+1}) = \begin{pmatrix} \widetilde{L}_{\vec{h};1,1}(x, \vec{\xi}) & \widetilde{L}_{\vec{h};1,2}(x, \vec{\xi}) \\ \widetilde{L}_{\vec{h};2,1}(x, \vec{\xi}) & \widetilde{L}_{\vec{h};2,2}(x, \vec{\xi}) \end{pmatrix} , \quad \xi_L := \xi_0 .$$

We consider the variables $(x, \xi_0, \ldots, \xi_{L-1})$ in the following spectral curve,

$$\mathcal{C}_{\vec{h}}: \quad \xi_j^N = (-1)^N \frac{\xi_{j+1}^N a_j^N - x^N b_j^N}{\xi_{j+1}^N x^N c_j^N - d_j^N} \quad , \quad j = 0, \ldots, L-1, \tag{3}$$

and denote $p_j = (x, \xi_j, \xi_{j+1})$. Then the operator $F_{h_j}(x, \xi_j, \xi_j)$ has 1-dimensional null space in $\mathbf{C}^N$ generated by the vector $|p_j\rangle$ with the form:

$$\langle 0|p_j\rangle = 1 \ , \quad \frac{\langle m|p_j\rangle}{\langle m-1|p_j\rangle} = \frac{\xi_{j+1} a_j \omega^m - x b_j}{-\xi_j(\xi_{j+1} x c_j \omega^m - d_j)} \ .$$

The Baxter vector $|p\rangle$ for $p \in \mathcal{C}_{\vec{h}}$ is defined by $|p\rangle := |p_0\rangle \otimes \ldots \otimes |p_{L-1}\rangle \in \overset{L}{\otimes} \mathbf{C}^N$, which possesses the following property:

$$\widetilde{L}_{\vec{h};1,1}(x, \vec{\xi})|p\rangle = |\tau_- p\rangle \Delta_-(p), \ \widetilde{L}_{\vec{h};2,2}(x, \vec{\xi})|p\rangle = |\tau_+ p\rangle \Delta_+(p) \ , \ \widetilde{L}_{\vec{h};2,1}(x, \vec{\xi})|p\rangle = 0 \ ,$$

where $\Delta_\pm$ are functions of $\mathcal{C}_{\vec{h}}$ defined by $\Delta_-(x, \vec{\xi}) = \prod_{j=0}^{L-1}(d_j - x\xi_{j+1}c_j)$, $\Delta_+(x, \vec{\xi}) = \prod_{j=0}^{L-1} \frac{\xi_j(a_j d_j - x^2 b_j c_j)}{\xi_{j+1} a_j - x b_j}$, and $\tau_\pm$ are the automorphisms, $\tau_\pm(x, \vec{\xi}) = (q^{\pm 1}x, q^{-1}\vec{\xi})$. It follows the important relation of the transfer matrix on the Baxter vector over the curve $\mathcal{C}_{\vec{h}}$,

$$T_{\vec{h}}(x)|p\rangle = |\tau_- p\rangle \Delta_-(p) + |\tau_+ p\rangle \Delta_+(p) \ , \quad \text{for} \ \ p \in \mathcal{C}_{\vec{h}} \ . \tag{4}$$

As $T_{\vec{h}}(x)$ are commuting operators for $x \in \mathbf{C}$, a common eigenvector $\langle\varphi|$ is a constant vector of $\overset{L}{\otimes} \mathbf{C}^N$ with an eigenvalue $\Lambda(x) \in \mathbf{C}[x]$. Define the function $Q(p) = \langle\varphi|p\rangle$ of $\mathcal{C}_{\vec{h}}$, then it satisfies the following Bethe equation,

$$\Lambda(x)Q(p) = Q(\tau_-(p))\Delta_-(p) + Q(\tau_+(p))\Delta_+(p) \ , \quad \text{for} \ p \in \mathcal{C}_{\vec{h}} \ . \tag{5}$$

By the definition of $T_{\vec{h}}(x), \Lambda(x)$, one can easily see that $T_{\vec{h}}(x)$ is an operator-coefficient even x-polynomial of degree $2[\frac{L}{2}]$ with the constant term $T_0 = \prod_{j=0}^{L-1} a_j \overset{L}{\otimes} Y + \prod_{j=0}^{L-1} d_j$. Hence the polynomial $\Lambda(x)$ in (5) is an even function of degree $\leq 2[\frac{L}{2}]$ with $\Lambda(0) = q^l \prod_{j=0}^{L-1} a_j + \prod_{j=0}^{L-1} d_j$ for some $l \in \mathbf{Z}_N$. For $L = 3$, we have $T_{\vec{h}}(x) = T_0 + x^2 T_2$ where

$$\begin{aligned} T_2 = {} & b_0 c_1 a_2 X \otimes Z \otimes Y + a_0 b_1 c_2 Y \otimes X \otimes Z + c_0 a_1 b_2 Z \otimes Y \otimes X \\ & + c_0 b_1 d_2 Z \otimes X \otimes I + d_0 c_1 b_2 I \otimes Z \otimes X + b_0 d_1 c_2 X \otimes I \otimes Z \ . \end{aligned} \tag{6}$$

The above T_2 can be put into the form of the Hofstadter type Hamiltonian (1).[6,10]

In the equation (5), $Q(p)$ is a rational function of $\mathcal{C}_{\vec{h}}$ with zeros and poles. Hence the understanding of the Bethe solutions of (5) relies heavily on the function theory of $\mathcal{C}_{\vec{h}}$, and the algebraic geometry of the curve inevitably plays a key role on the complexity of the problem.

3. The Rational Degenerated Bethe Equation

In this section, we consider the case when the spectral curve $\mathcal{C}_{\vec{h}}$ degenerates into an union of rational curves under the conditions:$a_j = q^{-1}d_j$, $b_j = q^{-1}c_j$ for $j = 0, \ldots, L-1$. By replacing c_j, d_j by $\frac{c_j}{d_j}, 1$, we assume $d_j = 1$ for all j with the parameter c_js to be generic. In this case, $\mathcal{C}_{\vec{h}}$ is the union of disjoint copies of the x-(complex) line, containing the following $\tau_\pm$-invariant subset of $\mathcal{C}_{\vec{h}}$ which will be sufficient for the discussion of Bethe equation,

$$\mathcal{C} := \{(x, \xi_0, \ldots, \xi_{L-1}) | \xi_0 = \cdots = \xi_{L-1} = q^l, l \in \mathbf{Z}_N\}.$$

We shall make the identification $\mathcal{C} = \mathbf{P}^1 \times \mathbf{Z}_N$ via $(x, q^l, \ldots, q^l) \leftrightarrow (x, l)$. The automorphisms $\tau_\pm$ on $\mathcal{C}$ become $\tau_\pm(x, l) = (q^{\pm 1}x, l-1)$, by which the action (4) of $T(x)(:= T_{\vec{h}}(x))$ on the Baxter vector $|x, l\rangle$ now takes the form,

$$T(x)|x, l\rangle = |q^{-1}x, l-1\rangle\Delta_-(x, l) + |qx, l-1\rangle\Delta_+(x, l) \ , \tag{7}$$

where $\Delta_\pm$ are the rational functions of x:$\Delta_-(x, l) = \prod_{j=0}^{L-1}(1 - xc_jq^l)$, $\Delta_+(x, l) = \prod_{j=0}^{L-1}\frac{1-x^2c_j^2}{1-xc_jq^{-l}}$. Furthermore, one can express the Baxter vector $|x, l\rangle$ over the curve $\mathcal{C}$ in the component-form: $\langle \mathbf{k}|x, l\rangle = q^{|\mathbf{k}|^2}\prod_{j=0}^{L-1}\frac{(xc_jq^{-l-2};\omega^{-1})_{k_j}}{(xc_jq^{l+2};\omega)_{k_j}}$. Here the bold letter $\mathbf{k}$ denotes a multi-index vector $\mathbf{k} = (k_0, \ldots, k_{L-1})$ for $k_j \in \mathbf{Z}_N$ with the square-length of $\mathbf{k}$ defined by $|\mathbf{k}|^2 := \sum_{j=0}^{L-1}k_j^2$. Each ratio-term in the above right hand side is given by a non-negative representative for each element in $\mathbf{Z}_N$ appeared in the formula. We have the following result on the Bethe equation and its connection with the transfer matrix $T(x)$:

Theorem 1: Denote f^e, f^o the functions on $\mathcal{C}$, $f^e(x, 2n) = \prod_{j=0}^{L-1}\frac{(xc_j;\omega^{-1})_{n+1}}{(xc_j;\omega)_{n+1}}$, and $f^o(x, 2n+1) = \prod_{j=0}^{L-1}\frac{(xc_jq^{-1};\omega^{-1})_{n+1}}{(xc_jq;\omega)_{n+1}}$. For $x \in \mathbf{P}^1$, $l \in \mathbf{Z}_N$, we define the following vectors in $\overset{L}{\otimes}\mathbf{C}^N$,

$$|x\rangle_l^e = \sum_{n=0}^{N-1}|x, 2n\rangle f^e(x, 2n)\omega^{ln} \ , \quad |x\rangle_l^o = \sum_{n=0}^{N-1}|x, 2n+1\rangle f^o(x, 2n+1)\omega^{ln} \ ,$$

$$|x\rangle_l^+ = |x\rangle_l^e q^{-l}u(qx) + |x\rangle_l^o u(x); \quad \text{where } u(x) := \prod_{j=0}^{L-1}(1 - x^Nc_j^N)(xc_jq; q^2)_M \ .$$

Then

(i) $|x\rangle_l^e u(qx) = |x\rangle_l^o q^l u(x)$, or equivalently, $|x\rangle_l^+ = 2q^{-l}|x\rangle_l^e u(qx) = 2|x\rangle_l^o u(x)$.

(ii) The $T(x)$-transform on $|x\rangle_l^+$ is given by

$$q^{-l}T(x)|x\rangle_l^+ = |q^{-1}x\rangle_l^+\Delta_-(x, -1) + |qx\rangle_l^+\Delta_+(x, 0) \ , \quad l \in \mathbf{Z}_N \ .$$

(iii) For a common eigenvector $\langle\varphi|$ of $T(x)$ with the eigenvalue $\Lambda(x)$, the function $Q_l^+(x)(:= \langle\varphi|x\rangle_l^+)$ and $\Lambda(x)$ are polynomials with the properties: $\deg.Q_l^+(x) \leq (3M+1)L$, $\deg.\Lambda(x) \leq 2[\frac{L}{2}]$, $\Lambda(x) = \Lambda(-x)$, $\Lambda(0) = q^{2l} + 1$, and the following Bethe equation holds:

$$q^{-l}\Lambda(x)Q_l^+(x) = \prod_{j=0}^{L-1}(1 - xc_jq^{-1})Q_l^+(xq^{-1}) + \prod_{j=0}^{L-1}(1 + xc_j)Q_l^+(xq) \ . \tag{8}$$

Furthermore for $0 \leq m \leq M$, $Q_m^+(x), Q_{N-m}^+(x)$ are elements in $x^m \prod_{j=0}^{L-1}(1-x^N c_j^N)\,\mathbf{C}[x]$.

By (iii) of the above theorem, the equation (8) for the sector $m, N-m$ can be combined into a single one. For the rest of this report the letter m will always denote an integer between 0 and M: $0 \leq m \leq M$. By introducing the polynomials $\Lambda_m(x), Q(x)$ via the relation,

$$(\Lambda_m(x),\ x^m \prod_{j=0}^{L-1}(1-x^N c_j^N)Q(x)) = (q^{-m}\Lambda(x),\ Q_m^+(x)),\quad (q^m\Lambda(x),\ Q_{N-m}^+(x))\ ,$$

the equation (8) for $l = m, N-m$ becomes the following polynomial equation of $Q(x), \Lambda_m(x)$:

$$\Lambda_m(x)Q(x) = q^{-m}\prod_{j=0}^{L-1}(1-xc_jq^{-1})Q(xq^{-1}) + q^m \prod_{j=0}^{L-1}(1+xc_j)Q(xq)\ , \qquad (9)$$

with $\deg.Q(x) \leq ML-m$, $\deg.\Lambda_m(x) \leq 2[\frac{L}{2}]$, $\Lambda_m(x) = \Lambda_m(-x), \Lambda_m(0) = q^m + q^{-m}$. The general mathematical problem will be to determine the solution space of the Bethe equation (9) for a given positive integer L.

For $L = 1, 2$, we have the following result.

Theorem 2: (I) For $L = 1$, we have $\Lambda_m(x) = q^m + q^{-m}$ and the solutions $Q_m(x)$ of (9) form an one-dimensional vector space generated by the following polynomial of degree $M-m$,

$$B_m(x) = 1 + \sum_{j=1}^{M-m}(\prod_{i=1}^{j}\frac{q^{m+i-1}-q^{-m-i}}{q^m+q^{-m}-q^{-m-i}-q^{m+i}})(xc_0)^j\ .$$

(II) For $L = 2$, the equation (9) has a non-trivial solution $Q_m(x)$ if and only if $\deg.Q_m(x) = M-m+m'$ for $0 \leq m' \leq M$. For each such m', the eigenvalue $\Lambda_m(x)$ in (9) is equal to $\Lambda_{m,m'}(x) := q^{\frac{1}{2}}(q^{m'-1}+q^{-m'-2})x^2c_0c_1 + q^m + q^{-m}$, and the corresponding solutions of $Q_m(x)$ form an one-dimensional space generated by a polynomial $B_{m,m'}(x)$ of degree $M-m+m'$ with $B_{m,m'}(0) = 1$.

For $L = 3$, this is the case related to the Hamiltonian (1). We consider the $N \times N$ matrix,

$$A = \begin{pmatrix} \delta'_{N-1} & u'_{N-1} & 0 & \cdots & & 0 & 0 \\ v'_{N-2} & \delta'_{N-2} & u'_{N-2} & 0 & \ddots & & \vdots \\ w'_{N-3} & v'_{N-3} & \delta'_{N-3} & u'_{N-3} & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & w'_1 & v'_1 & \delta'_1 & u'_1 \\ 0 & \cdots & & 0 & w'_0 & v'_0 & \delta'_0 \end{pmatrix} \qquad (10)$$

with the entries defined by $w'_k = q^{k+\frac{3}{2}} + q^{-k-\frac{3}{2}} - q^m - q^{-m}$, $v'_k = (q^{k+\frac{1}{2}} - q^{-k-\frac{3}{2}})(c_0 + c_1 + c_2)$, $\delta'_k = (q^{k-\frac{1}{2}} + q^{-k-\frac{3}{2}})(c_0c_1 + c_1c_2 + c_2c_0)$, $u'_k = (q^{k-\frac{3}{2}} - q^{-k-\frac{3}{2}})c_0c_1c_2$. Then one can derive the following result.

Theorem 3: For $L = 3$, the condition of the eigenvalue $\Lambda_m(x) = \lambda_m x^2 + q^m + q^{-m}$, $0 \leq m \leq M$, with a non-trivial solution $Q_m(x)$ in the equation (9) is determined by the solution of $\det(A - \lambda_m) = 0$, where A is the matrix defined by (10). For each such $\Lambda_m(x)$, there exists an unique (up to constants) non-trivial polynomial solution $Q_m(x)$ of (9) with the degree $Q_m(x)$ equal to $3M - m$ and $Q_m(0) \neq 0$.

4. The Degeneracy and Bethe Ansatz Relation of Roots of Bethe Polynomial

We first discuss the degeneracy relation of eigenspaces of the transform matrix $T(x)$ in $\overset{L}{\otimes}\ \mathbf{C}^{N*}$ with respect to the Bethe solutions obtained in the previous section. As before, we denote $\Lambda(x)$ the eigenvalues of $T(x)$, whose constant term is given by $T_0 = D + 1$, where $D := q^{-L} \overset{L}{\otimes} Y$; hence $\Lambda(0) = q^l + 1$. For $l \in \mathbf{Z}_N$, we denote $\mathbf{E}_L^l$ the N^{L-1}-dimensional eigensubspace of $\overset{L}{\otimes}\ \mathbf{C}^{N*}$ of the operator D with the eigenvalue q^l. For $0 \leq m \leq M$, the equation (9) describes the relation of $\Lambda(x)$ and its eigenfunctions with $\Lambda(0) = q^{2m} + 1$ or $q^{2(N-m)} + 1$. We now consider the case for $L = 3$, where T_2 in (6) is now expressed by

$$\begin{aligned} T_2 = {} & q^{-2}(c_0c_1 X \otimes Z \otimes Y + c_1c_2 Y \otimes X \otimes Z + c_0c_2 Z \otimes Y \otimes X) \\ & + q^{-1}(c_0c_1 Z \otimes X \otimes I + c_1c_2 I \otimes Z \otimes X + c_0c_2 X \otimes I \otimes Z). \end{aligned}$$

We have $qD = (Z \otimes X \otimes I)(X \otimes I \otimes Z)(I \otimes Z \otimes X)$. We shall denote $\mathcal{O}_3$ the operator algebra generated by the tensors of X, Y, Z, I appeared in the above expression of T_2. Then $\mathcal{O}_3$ commutes with D, hence one obtains a $\mathcal{O}_3$-representation on $\mathbf{E}_3^l$ for each l. With the identification, $U = D^{-1/2} Z \otimes X \otimes I$, $V = D^{-1/2} X \otimes I \otimes Z$, $\mathcal{O}_3$ is generated by U, V which satisfy the Weyl relation $UV = \omega VU$ and the N-th power identity. Hence $\mathcal{O}_3$ is the Heisenberg algebra and contains D as a central element. Then $qD^{\frac{-1}{2}} T_2$ has the following expression,

$$c_0c_1(U + U^{-1}) + c_0c_2(V + V^{-1}) + c_1c_2(qD^{5/2}UV + q^{-1}D^{-5/2}V^{-1}U^{-1}) \,. \quad (11)$$

The above Hamiltonian is the same as H_{FK} (1) with $W = q^{-1}D^{-5/2}V^{-1}U^{-1}$, $\alpha = \beta = \gamma = 1$. Our conclusion on the sector $m = M$ is equivalent to that in Ref. 6 as it becomes clearer later on. There is an unique (up to equivalence) non-trivial irreducible representation of $\mathcal{O}_3$, denoted by $\mathbf{C}_\rho^N$, which is of dimension N. For each l, $\mathbf{E}_3^l$ is equivalent to N-copies of $\mathbf{C}_\rho^N$ as $\mathcal{O}_3$-modules: $\mathbf{E}_3^l \simeq N\,\mathbf{C}_\rho^N$. For $0 \leq m \leq M$, we consider the space $\mathbf{E}_3^l$ with $q^l = q^{\pm 2m}$. The evaluation of $\mathbf{E}_3^l$ on $|x\rangle_{\pm m}^{+}$ gives rise to a N-dimensional kernel in $\mathbf{E}_3^l$. By Theorem 3, there are N polynomials $Q_m(x)$ of degree $3M - m$ as solutions of (9) with the corresponding N distinct eigenvalues $\Lambda_m(x)$. The N-dimensional vector space spanned by those $Q_m(x)$s becomes a realization of the irreducible representation $\mathbf{C}_\rho^N$ for the Heisenberg algebra $\mathcal{O}_3$.

Now we discuss the relation between the Bethe equation (9) and the usual Bethe ansatz formulation in literature. For $0 \leq m \leq M$, a solution $Q_m(x)$ in (9) always have the property $Q_m(0) \neq 0$ by Theorem 3, hence one has the form $\alpha_{3M-m}^{-1} Q_m(x) = \prod_{l=1}^{3M-m}(x - \frac{1}{z_l})$ with $z_l \in \mathbf{C}^*$. By setting $x = z_l^{-1}$ in (9), we obtain the following relation among z_ls, which is called the Bethe ansatz relation,

$$q^{m+\frac{3}{2}} \prod_{j=0}^{2} \frac{z_l + c_j}{qz_l - c_j} = \prod_{n=1, n\neq l}^{3M-m} \frac{qz_l - z_n}{z_l - qz_n}, \ 1 \leq l \leq 3M - m \ .$$

For the sector $m = M$, the comparison of the x^2-coefficients of (9) yields the expression of eigenvalue,

$$\lambda_M = (q^{\frac{-1}{2}} + q^{\frac{-3}{2}})s_2 + (q^{\frac{1}{2}} - q^{\frac{-3}{2}})s_1 \sum_{n=1}^{2M} z_n + (q^{\frac{3}{2}} + q^{\frac{-3}{2}} - q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \sum_{l<n} z_l z_n \ .$$

With the substitution, $\mu = q^{\frac{1}{2}} c_0^{-1}, \nu = q^{\frac{1}{2}} c_1^{-1}, \rho = q^{\frac{1}{2}} c_2^{-1}$, the above expression coincides with (5.27) in Ref. 6. Note that the Bethe ansatz relation can be shown to be equivalent to the Bethe equation (9) for the sector M. However, the parallel statement is no longer true for other sectors $m \neq M$, i.e., it does exist some non-physical Bethe ansatz solutions in the above form, while not corresponding to any polynomial solution of Bethe equation (9). Some example can be found in the $(M-1)$-sector.

5. High Genus Curves for the Hofstadter Model

We are now going back to the general situation in Sect. 2. Note that the values ξ_j^Ns of the curve $\mathcal{C}_{\vec{h}}$ in (3) are determined by ξ_0^N and x^N, denoted by $y = x^N$, $\eta = \xi_0^N$. The variables (y, η) defines the curve which is a double cover of y-line,

$$\mathcal{B}_{\vec{h}}: \ \ C_{\vec{h}}(y)\eta^2 + (A_{\vec{h}}(y) - D_{\vec{h}}(y))\eta - B_{\vec{h}}(y) = 0$$

where the functions $A_{\vec{h}}, B_{\vec{h}}, C_{\vec{h}}, D_{\vec{h}}$ are the following matrix elements,

$$\begin{pmatrix} -A_{\vec{h}}(y) & B_{\vec{h}}(y) \\ C_{\vec{h}}(y) & -D_{\vec{h}}(y) \end{pmatrix} := \prod_{j=0}^{L-1} \begin{pmatrix} -a_j^N & yb_j^N \\ yc_j^N & -d_j^N \end{pmatrix} \ .$$

Now we consider only the case: $L = 3$, $a_0 = d_0 = 0, b_0 = c_0 = 1$, with generic h_1, h_2. The expression of $T(x)$ is given by

$$T(x) = x^2(c_1 a_2 X \otimes Z \otimes Y + a_1 b_2 Z \otimes Y \otimes X + b_1 d_2 Z \otimes X \otimes I + d_1 c_2 X \otimes I \otimes Z),$$

equivalently, $x^{-2} D^{\frac{-1}{2}} T(x)$ is equal to the Hofstadter Hamiltonian $(1)_{\rho=0}$ with $U = D^{-1/2} Z \otimes X \otimes I$, $V = D^{-1/2} X \otimes I \otimes Z$ and μ, ν, α, β related to h_1, h_2 by $\mu^2 = qb_1 c_1 a_2 d_2$, $\alpha^2 = q^{-1} b_1 c_1^{-1} a_2^{-1} d_2$, $\nu^2 = q a_1 d_1 b_2 c_2$, $\beta^2 = q^{-1} a_1^{-1} d_1 b_2^{-1} c_2$. By factoring out the y-component of $\mathcal{B}_{\vec{h}}$, the main irreducible component of $\mathcal{B}_{\vec{h}}$ is the curve,

$$\mathcal{B}: \ (y^2 b_1^N c_2^N + a_1^N a_2^N)\eta^2 + (a_1^N b_2^N + b_1^N d_2^N - c_1^N a_2^N - d_1^N c_2^N) y\eta - (y^2 c_1^N b_2^N + d_1^N d_2^N) = 0 \, ,$$

which is an elliptic curve as a double-cover of the y-line. For the curve $\mathcal{C}_{\vec{h}}$, the variables ξ_0 and ξ_1 are related by $\xi_0^N = \xi_1^{-N}$, which implies that $\mathcal{C}_{\vec{h}}$ can be identified with $\mathcal{W} \times \mathbf{Z}_N$ where $\mathcal{W}$ is a genus $6N^3 - 6N^2 + 1$ curve with the following equation in the variable $p = (x, \xi_0, \xi_2)$,

$$\mathcal{W}: \quad \xi_0^{-N} = \frac{-\xi_2^N a_1^N + x^N b_1^N}{x^N \xi_2^N c_1^N - d_1^N}, \quad \xi_2^N = \frac{-\xi_0^N a_2^N + x^N b_2^N}{x^N \xi_0^N c_2^N - d_2^N}.$$

By averaging the Baxter vectors $|p, s\rangle$ of $\mathcal{C}_{\vec{h}}$ over an element p of $\mathcal{W}$, $|p\rangle := \frac{1}{N}\sum_{s=0}^{N-1} |p, s\rangle q^{s^2}$, which defines the Baxter vector on $\mathcal{W}$. Furthermore, the transfer matrix can be descended to one on $\mathcal{W}$ with the following relation,

$$x^{-2} T(x)|p\rangle = |\tau_-(p)\rangle \widetilde{\Delta}_-(p) + |\tau_+(p)\rangle \widetilde{\Delta}_+(p) ,$$

where $\widetilde{\Delta}_\pm$ are the functions on $\mathcal{W}$: $\widetilde{\Delta}_-(x, \xi_0, \xi_2) = \frac{(x\xi_2 c_1 - d_1)(x\xi_0 c_2 - d_2)}{-x\xi_0}$, $\widetilde{\Delta}_+(x, \xi_0, \xi_2) = \frac{\xi_2 (a_1 d_1 - x^2 b_1 c_1)(a_2 d_2 - x^2 b_2 c_2)}{x(\xi_2 a_1 - x b_1)(\xi_0 a_2 - x b_2)}$. For an eigenvector $\langle\varphi| \in \overset{3}{\otimes} \mathbf{C}^{N*}$ of $x^{-2}T(x)$, the eigenvalue is a scalar $\lambda \in \mathbf{C}$, and the function $Q(p) := \langle\varphi|p\rangle$ of $\mathcal{W}$ satisfies the Bethe equation: $\lambda Q(p) = Q(\tau_-(p))\widetilde{\Delta}_-(p) + Q(\tau_+(p))\widetilde{\Delta}_+(p)$, where $\tau^\pm$ are the transformations of $\mathcal{W}$ with the same expression as before, but only in the coordinates (x, ξ_0, ξ_2). Consider the D-eigenspace decomposition of $\overset{3}{\otimes} \mathbf{C}^{N*} = \bigoplus_{l \in \mathbf{Z}_N} \mathbf{E}_3^l$. The evaluation of the Baxter vector over $\mathcal{W}$ gives rise to the following linear transformation, $\varepsilon_l : \mathbf{E}_3^l \longrightarrow \{\text{rational functions of } \mathcal{W}\}$ with $\varepsilon_l(v)(p) := \langle v|p\rangle$, for $l \in \mathbf{Z}_N$. One has the following result.

Theorem 4: For $l \in \mathbf{Z}_N$, the linear map ε_l is injective, hence it induces an identification of $\mathbf{E}_3^l$ with a N^2-dimensional functional space of $\mathcal{W}$.

By the discussion in Sect. 4, as the Heisenberg algebra $\mathcal{O}_3$ representations , $\mathbf{E}_3^l$ is equivalent to N copies of the standard one. Hence it induces an $\mathcal{O}_3$-module structure on the function space $\varepsilon_l(\mathbf{E}_3^l)$, induced by the one of $\mathbf{E}_3^l$ by above theorem. The mathematical structure of the functional space $\varepsilon_l(\mathbf{E}_3^l)$ in terms of the divisor theory of Riemann surfaces in corporation with the interpretation of Heisenberg algebra representation remains an algebraic geometry problem for further study.

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LIMITATIONS ON QUANTUM CONTROL

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In this note we give an introduction to the topic of Quantum Control, explaining what its objectives are, and describing some of its limitations.

1. What is Quantum Control?

The objectives of Quantum Control are

- To determine quantum mechanical systems which will drive an initial given state to a pre-determined final state, the target state.
- To describe—and hopefully implement—quantum systems which will through time evolution optimize given operator expectations corresponding to observables of the system.

Among a wealth of applications are those to Quantum Computing, where it is clearly essential to be able to start off a quantum procedure with a given initial state, and to problems involving the population levels in atomic systems, such as the laser cooling of atomic or molecular systems.

The mathematical tools necessary for the theoretical investigation of these control problems are diverse, involving algebraic, group theoretic and topological methods.

The questions that one may ask include:

(i) When is a given quantum system completely controllable?
(ii) If a system is not completely controllable, how does this affect optimization of a given operator?
(iii) How near can you get to a target state for a not completely controllable system?

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The answer to the first question depends on a knowledge of the Lie algebra generated by the system's quantum hamiltonian, that to the second arises from properties of the Lie group structure, while the last clearly involves ideas of topology.

Especially in the area of Lie group theory, there is a large corpus of classical mathematics which can supply answers to questions arising in quantum control. In particular, for the type of controllability known as *Pure State Controllability* classical Lie Group theory has already given the basic results.

The quantum control system we shall consider is typically of the form

$$H = H_0 + \sum_{m=1}^{M} f_m(t) H_m, \tag{1}$$

where H_0 is the internal Hamiltonian of the unperturbed system and H_m are interaction terms governing the interaction of the system with an external field. The dynamical evolution of the system is governed by the unitary evolution operator $U(t,0)$, which satisfies the Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} U(t,0) = HU(t,0) \tag{2}$$

with initial condition $U(0,0) = I$, where I is the identity operator. By use of the Magnus expansion, it can be shown that the solution U involves all the commutators of the H_m. The operators H_m, $0 \leq m \leq M$, in (1) are Hermitian. Their skew-Hermitian counterparts iH_m generate a Lie algebra L known as the dynamical Lie algebra of the control system which is always a subalgebra of $u(N)$, or for trace-zero hamiltonians, $su(N)$. The degree of controllability is determined by the dynamical Lie algebra generated by the control system hamiltonian H. If $L = u(N)$ then *all* the unitary operators are generated and we call such a system *Completely Controllable.* A large variety of common quantum systems can be shown to be Completely Controllable.[1,2] The interesting cases arise when L is a proper subalgebra of $u(N)$. Such systems may still exhibit *Pure State Controllability,* in that starting with any initial pure state any target pure state may be obtained, as distinct from the Completely Controllable case, when all (kinematically admissible) states—pure or mixed—may be achieved.

2. Pure state controllability

We shall restrict our attention here to finite-level quantum systems with N discrete energy levels. The pure quantum states of the system are represented by normalized wavefunctions $|\Psi\rangle$, which form a Hilbert space H. However, the state of a quantum system need not be represented by a pure state $|\Psi\rangle \in H$. For instance, we may consider a system consisting of a large number of identical, non-interacting particles, which can be in different internal quantum states, i.e., a certain fraction w_1 of the particles may be in quantum state $|\Psi_1\rangle$, another fraction w_2 may be in another state $|\Psi_2\rangle$ and so forth. Hence, the state of the system as a whole is described by a discrete ensemble of quantum states $|\Psi_n\rangle$ with non-negative weights w_n that sum

up to one. Such an ensemble of quantum states is called a *mixed-state*, and it can be represented by a density operator ρ_0 on H with the spectral representation

$$\rho_0 = \sum_{n=1}^{N} w_n |\Psi_n\rangle\langle\Psi_n|, \tag{3}$$

where $\{|\Psi_n\rangle : 1 \leq n \leq N\}$ is an orthonormal set of vectors in H that forms an ensemble of independent pure quantum states. The evolution of ρ_0 is governed by

$$\rho(t) = U(t,0)\rho_0 U(t,0)^\dagger, \tag{4}$$

with $U(t,0)$ as above. Clearly if *all* the unitary operators can be generated we have the optimal situation, complete controllability. However, classical Lie group theory tells us that even when we only obtain a subalgebra of $u(N)$ we can obtain pure state controllability.

The results arise from consideration[3] of the transitive action of Lie groups on the sphere S^k. The classical "orthogonal" groups $\Theta(n,F)$ where the field F is either the reals $\Re$, the complexes $\mathcal{C}$ or the quaternions $\mathcal{H}$, are defined to be those that keep invariant the length of the vector $v \equiv (v_1, v_2, \ldots, v_n)$; the squared length is given by $v^\dagger v = \sum_{i=1}^{n} \bar{v}_i v_i$, where $\bar{v}_i$ refers to the appropriate conjugation. These compact groups are, essentially, the only ones which give transitive actions on the appropriate spheres, as follows:

(i) $\Theta(n,\Re) \equiv O(n)$ transitive on $S^{(n-1)}$
(ii) $\Theta(n,\mathcal{C}) \equiv U(n)$ transitive on $S^{(2n-1)}$
(iii) $\Theta(n,\mathcal{H}) \equiv Sp(n)$ transitive on $S^{(4n-1)}$.

Since we may regard our pure state as a normalized vector in $\mathcal{C}^N$ and thus as a point on $S^{(2N-1)}$, we obtain pure state controllability only for $U(N)$ (or $SU(N)$ if we are not too fussy about phases) and $Sp(N/2)$, the latter for even N only. (Note that we cannot get $O(2N)$ as a subalgebra of $U(N)$.)

Complete controllability is clearly a stronger condition than pure state controllability. To illustrate our theme of the limitations on quantum control, we now give two examples based on a truncated oscillator with nearest-level interactions for which the algebras generated are so(N) and sp(N/2). Both these examples are generic.

3. Examples

3.1. *Three-level oscillator with dipole interactions.*

Consider a three-level system with energy levels E_1, E_2, E_3 and assume the interaction with an external field f_1 is of dipole form with nearest neighbor interactions only. Then we have $H = H_0 + f(t)H_1$, where the matrix representations of H_0 and

H_1 are

$$H_0 = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0 & d_1 & 0 \\ d_1 & 0 & d_2 \\ 0 & d_2 & 0 \end{bmatrix}.$$

If the energy levels are equally spaced, i.e., $E_2 - E_1 = E_3 - E_2 = \mu$ and the transition dipole moments are equal, i.e., $d_1 = d_2 = d$ then we have

$$H_0' = \mu \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad H_1 = d \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

where H_0' is the traceless part of H_0. Both $\mathrm{i}H_0'$ and $\mathrm{i}H_1$ satisfy

$$A + A^\dagger = 0 \qquad AJ + J\tilde{A} = 0$$

where

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which is a defining relation for $so(3)$. The dynamical Lie algebra in this case is in fact $so(3)$. It is easy to show that the matrix $B = UAU^\dagger$ is a real anti-symmetric representation of $so(3)$ if $U = U^*J$. Explicitly, a suitable unitary matrix is given by

$$U \equiv \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & i & 0 \end{bmatrix}.$$

Since the dynamical algebra and group in the basis determined by U consists of *real* matrices, real states can only be transformed to real states; this means that for any initial state there is a large class of unreachable states. This example is generic as it applies to N-level systems, although for even N we need other than dipole interactions to generate $so(N)$. The analogous dipole interaction generates sp(N/2) in the even N case, as we now illustrate.

3.2. *Four-level oscillator with dipole interactions.*

Consider a four-level system with Hamiltonian $H = H_0 + f(t)H_1$,

$$H_0 = \begin{bmatrix} -E_1 & 0 & 0 & 0 \\ 0 & -E_2 & 0 & 0 \\ 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & E_1 \end{bmatrix},$$

and

$$H_1 = \begin{bmatrix} 0 & d_1 & 0 & 0 \\ d_1 & 0 & d_2 & 0 \\ 0 & d_2 & 0 & -d_1 \\ 0 & 0 & -d_1 & 0 \end{bmatrix}.$$

Note that $\mathrm{i}H_0$ and $\mathrm{i}H_1$ satisfy

$$x = -x^\dagger, \quad x^T J + Jx = 0 \tag{5}$$

for

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

where J is unitarily equivalent to

$$\left[\begin{array}{c|c} 0 & I_{N/2} \\ \hline -I_{N/2} & 0 \end{array}\right], \tag{6}$$

which is a defining relation for $sp(N/2)$. Consider an initial state of the form

$$\rho_0 = \mathrm{i}x + \alpha I_N, \tag{7}$$

where x satisfies (5), it can only evolve into states

$$\rho_1 = \mathrm{i}y + \alpha I_N, \tag{8}$$

where y satisfies (5), under the action of a unitary evolution operator in exponential image of L. Hence, any target state that is not of the form (8) is not accessible from the initial state (7). Note that the initial state

$$\rho_0 = \begin{bmatrix} 0.35 & 0 & 0 & 0 \\ 0 & 0.30 & 0 & 0 \\ 0 & 0 & 0.20 & 0 \\ 0 & 0 & 0 & 0.15 \end{bmatrix}$$

is of the form $\rho_0 = x + 0.25 I_4$ and that

$$\mathrm{i}x = \mathrm{i} \begin{bmatrix} 0.10 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \\ 0 & 0 & -0.05 & 0 \\ 0 & 0 & 0 & -0.10 \end{bmatrix}$$

satisfies (5). Consider the target state

$$\rho_1 = \begin{bmatrix} 0.30 & 0 & 0 & 0 \\ 0 & 0.35 & 0 & 0 \\ 0 & 0 & 0.20 & 0 \\ 0 & 0 & 0 & 0.15 \end{bmatrix}$$

which is clearly kinematically admissible since

$$\rho_1 = \begin{bmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \rho_0 \begin{bmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}^\dagger$$

but note that $\rho_1 = y + 0.25 I_4$ and

$$\mathrm{i}y = \mathrm{i} \begin{bmatrix} 0.05 & 0 & 0 & 0 \\ 0 & 0.10 & 0 & 0 \\ 0 & 0 & -0.05 & 0 \\ 0 & 0 & 0 & -0.10 \end{bmatrix}$$

does not satisfy (5). Hence, ρ_1 is not dynamically accessible from ρ_0 for this system.

Given a target state ρ_1 that is not dynamically accessible from an initial state ρ_0, we can easily construct observables whose kinematical upper bound for its expectation value can not be reached dynamically. Simply consider $A = \rho_1$. The expectation value of A assumes its kinematical maximum only when the system is in state ρ_1. Since ρ_1 is not reachable, the kinematical upper bound is not dynamically realizable.

4. Conclusions

In this short introduction to Quantum Control theory, we have described the goals of the subject briefly, and then illustrated the limitations by generic examples where complete control is not possible. The tools we have used are, in the main, those of classical Lie group theory. Theoretical problems that remain to be tackled include to what extent these non-controllable systems can, in fact, be controlled; and, of course, the paramount problem of implementing these controls in practice.

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RIGOROUS RESULTS ON THE STRONGLY CORRELATED ELECTRON SYSTEMS BY THE SPIN-REFLECTION-POSITIVITY METHOD *

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In this talk, we shall briefly review some results on the strongly correlated electron systems, derived recently by applying Lieb's spin-reflection-positivity method. To explain the basic ideas of this method to a wide audience, we emphasize the important role played by Marshall's rule in studying the many-body systems

Keywords: Strongly Correlated Electron Systems, Marshall's Rule, Spin-reflection-positivity Method

In the past several decades, the strongly correlated electron systems attract many physicists' interest. In particular, interplay between the itinerant magnetic orderings and the quantum transport properties of these systems is a main focus of the current research. To get insight into the strongly correlated electron materials, various models have been introduced and investigated. The best known examples are the Hubbard model,[1] the periodic Anderson model,[2] and the Kondo lattice model.[3]

As a concrete example, let us consider the Hubbard model. On a lattice Λ with N_Λ sites, the Hamiltonian of the Hubbard model has the following form

$$H_H = -t \sum_{\sigma} \sum_{<\mathbf{ij}>} \left(\hat{c}^{\dagger}_{\mathbf{i}\sigma} \hat{c}_{\mathbf{j}\sigma} + \hat{c}^{\dagger}_{\mathbf{j}\sigma} \hat{c}_{\mathbf{i}\sigma} \right) + U \sum_{\mathbf{i}\in\Lambda} \left(\hat{n}_{\mathbf{i}\uparrow} - \frac{1}{2} \right) \left(\hat{n}_{\mathbf{i}\downarrow} - \frac{1}{2} \right) - \mu \hat{N} , \quad (1)$$

where $\hat{c}^{\dagger}_{\mathbf{i}\sigma}$ ($\hat{c}_{\mathbf{i}\sigma}$) denotes the fermion creation (annihilation) operator which creates (annihilates) an electron of spin σ at lattice site $\mathbf{i}$. $< \mathbf{ij} >$ is a pair of lattice sites. $t > 0$ and $U > 0$ are parameters representing the hopping energy and the on-site Coulomb repulsion of electrons, respectively. In the following, we shall assume that, in terms of the Hamiltonian, lattice Λ is bipartite. In other words, it can be separated into two sublattices A and B, and electrons hops only from a site $\mathbf{i}$ in one sublattice to a site $\mathbf{j}$ in another sublattice.

*Dedicated to Professor F. Y. Wu's 70th birthday.

These models enjoy some symmetries, which can be exploited to simplify their analysis. For instance, the Hubbard Hamiltonian H_H commutes with the total particle number operator $\hat{N} = \sum_{\mathbf{i}}(\hat{n}_{\mathbf{i}\uparrow}+\hat{n}_{\mathbf{i}\downarrow})$. Therefore, the Hilbert space of this model can be divided into numerous subspaces $\{V(N)\}$. Each of them is characterized by a conserved particle number N. H_H also commutes with the following spin operators

$$\hat{S}_+ = \sum_{\mathbf{i}\in\Lambda} \hat{c}^\dagger_{\mathbf{i}\uparrow}\hat{c}_{\mathbf{i}\downarrow}, \quad \hat{S}_- = \sum_{\mathbf{i}\in\Lambda} \hat{c}^\dagger_{\mathbf{i}\uparrow}\hat{c}_{\mathbf{i}\downarrow}, \quad \hat{S}_z = \frac{1}{2}\sum_{\mathbf{i}\in\Lambda}(\hat{n}_{\mathbf{i}\uparrow} - \hat{n}_{\mathbf{i}\downarrow}) \,. \tag{2}$$

Consequently, both S^2 and S_z are good quantum numbers.

Furthermore, when $\mu = 0$, the ground state of the Hubbard Hamiltonian in the half-filled subspace with $N = N_\Lambda$ is actually its global ground state. In this case, the Hubbard Hamiltonian has another symmetry: The pseudospin spin symmetry. More precisely, H_H commutes with operators

$$\hat{J}_+ = \sum_{\mathbf{i}\in\Lambda} \epsilon(\mathbf{i})\hat{c}^\dagger_{\mathbf{i}\uparrow}\hat{c}^\dagger_{\mathbf{i}\downarrow}, \quad \hat{J}_- = \sum_{\mathbf{i}\in\Lambda} \epsilon(\mathbf{i})\hat{c}_{\mathbf{i}\downarrow}\hat{c}_{\mathbf{i}\uparrow}, \quad \hat{J}_z = \frac{1}{2}\sum_{\mathbf{i}\in\Lambda}(\hat{n}_{\mathbf{i}\uparrow} + \hat{n}_{\mathbf{i}\downarrow} - 1)\,, \tag{3}$$

where $\epsilon(\mathbf{i}) = 1$ for $\mathbf{i} \in A$ and $\epsilon(\mathbf{i}) = -1$ for $\mathbf{i} \in B$. In literature, these operators are called the pseudospin operators.[4]

In one dimension, the Hubbard model can be exactly solved by applying the Bethe ansatz.[5] However, in higher dimensions, this approach fails. Instead, various approximate analytical techniques were introduced and developed. Most of them are based on the mean-field theories.[6] By applying effectively these methods, many interesting results on the strongly correlated electron systems have been derived. On the other hand, if it is possible, one would naturally like to re-establish some of these conclusions on a more rigorous basis. It will deepen our understanding on the electronic correlations in these models.

In a seminal paper published in 1989, Lieb introduced a powerful method, the spin-reflection-positivity technique, to investigate the Hubbard Hamiltonian with an even number of electrons.[7] With this method, Lieb proved that the ground state of the Hubbard Hamiltonian at half-filling is nondegenerate and has total spin $S = (1/2)|N_A - N_B|$, where N_A and N_B are the numbers of the sites in sublattice A and B, respectively. Later, this technique was also applied to both the periodic Anderson model and the Kondo lattice model.[8,9] Similar conclusions were established.

Later, we applied this method to investigating the spin and the off-diagonal correlation functions, as well as the excitation gaps of these strongly correlated electron models. In a series of papers, we proved that

- The on-site pairing correlation function of the negative-U Hubbard model is nonnegative.[10] More precisely, for any pair of lattice sites $\mathbf{h}$ and $\mathbf{k}$, inequality

$$\langle \Psi_0(-U) \mid \hat{c}^\dagger_{\mathbf{h}\uparrow}\hat{c}^\dagger_{\mathbf{h}\downarrow}\hat{c}_{\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\uparrow} \mid \Psi_0(-U)\rangle \geq 0 \tag{4}$$

holds true for the ground state of the negative-U Hubbard model in $V(2M)$. This inequality confirms that the Bose-Einstein condensation in the negative-U

Hubbard model occurs at zero-momentum.

- At half-filling, the spin correlations in the ground states of the periodic Anderson model and the Kondo lattice model are antiferromagnetic.[11] Very recently, by extending the spin-reflection-positivity method to the case of nonzero temperature, we also proved that, at any $T \neq 0$, the antiferromagnetic spin correlation in these models is dominant.[12]
- On some special lattices subject to condition $|N_A - N_B| = O(N_\Lambda)$, such as the example shown in Fig. 1, the ground state of the Hubbard model has both the antiferromagnetic and ferromagnetic long-range orders. In other words, it is a ferrimagnet.[13]
- Define $\Delta_{qp} = E_0(N_\Lambda + 1) + E_0(N_\Lambda - 1) - 2E_0(N_\Lambda)$ and $\Delta_S = E_0(N_\Lambda, S = 1) - E_0(N_\Lambda, S = 0)$ to be the quasi-particle gap and the spin excitation gap of the Hubbard model, respectively. Then, relation $\Delta_{qp} \geq \Delta_S$ holds true. Similar inequalities were also proven for the periodic Anderson model and the Kondo lattice model.[14]
- Define $\Delta_C = E_0(J = 1) - E_0(J = 0) = E_0(N_\Lambda + 2) - E_0(N_\Lambda)$ to be the charged gap of these strongly correlated electron models, then inequality $\Delta_C \geq \Delta_S$ holds.[15]

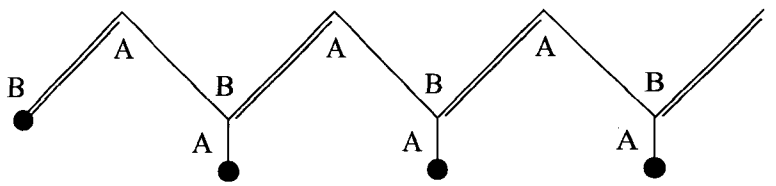

Fig. 1. The lattice structure of organic conjugated polymers.

Due to the page limit of this paper, it is impossible to discuss the spin-reflection-positivity method and its applications in details. In the following, we shall briefly explain the basic ideas of it. As a matter of fact, this method is closely related to a very simple but important observation: *After a proper unitary transformation, the ground state of a realistic quantum many-body Hamiltonian, in general, satisfies Marshall's rule,*[16] *in a sense.*

To begin with, let us first consider a simple quantum mechanical system: One particle moving in a one-dimensional well, as shown in Fig. 2. The ground state $\Psi_0(x)$ of this system satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi_0(x)}{dx^2} + v(x)\Psi_0(x) = E_0\Psi_0(x)\,. \tag{5}$$

When $v(x) = 0$, $\Psi_0(x)$ can be explicitly solved and we have $\Psi_0(x) = \sqrt{2/R}\sin(\pi x/R) > 0$ for any $0 < x < R$. This is Marshall's rule for Ψ_0 in this special case. A direct corollary of this rule is that the ground state $\Psi_0(x)$ is nondegenerate.

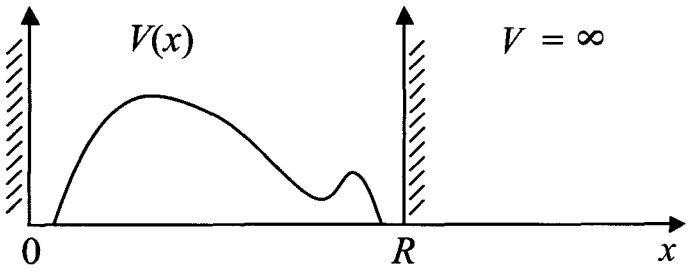

Fig. 2. The potential function of a one-dimensional quantum well.

To show that $\Psi_0(x)$ also satisfies the Marshall's rule when $v(x) \neq 0$, we notice that $\Psi_0(x)$ is a ground-state solution of Eq. (5), if and only if it is also a minimizing function of the following energy functional

$$\mathcal{E}(\psi) = \frac{\hbar^2}{2m} \int_0^R \left| \frac{d\psi(x)}{dx} \right|^2 dx + \int_0^R |\psi(x)|^2 v(x) dx \tag{6}$$

in some function space $H^1(0, R)$, requiring that both $|\psi|^2$ and $|d\psi/dx|^2$ are integrable over $(0, R)$.

Assume that $\Psi_0(x)$ has indefinite sign in the interval, as shown in Fig. 3. Then,

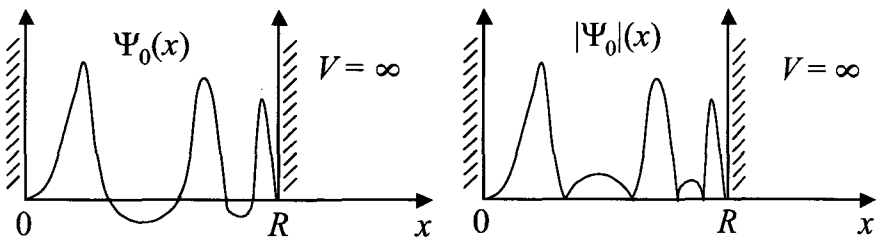

Fig. 3. The ground state wave functions $\Psi_0(x)$ and $|\Psi_0|(x)$.

we construct a new function $|\Psi_0|(x)$ by taking the absolute value of $\Psi_0(x)$. We notice that replacement of $\Psi_0(x)$ with $|\Psi_0|(x)$ does not change the value of the second term in Eq. (6). On the other hand, it can be shown

$$\int_0^R \left| \frac{d\Psi_0(x)}{dx} \right|^2 dx \geq \int_0^R \left| \frac{d \mid \Psi_0 \mid (x)}{dx} \right|^2 dx \tag{7}$$

(See the appendix of Ref. 17). Consequently, we have

$$\min_{\psi \in H^1} \mathcal{E}(\psi) = \mathcal{E}(\Psi_0) \geq \mathcal{E}(\mid \Psi_0 \mid) \tag{8}$$

In other words, $\mid \Psi_0 \mid (x)$ must be also a minimizing function of energy functional (6) and hence, a ground state solution of Eq. (5).

Next, we show that the ground state wave function $\Psi_0(x)$ has no zero point in $(0, R)$. If this is not true, then, there must be, at least, one point $x_0 \in (0, R)$ such that $|\Psi_0|(x_0) = 0$. However, as a nonnegative solution of a second order elliptical differential equation, $|\Psi_0|(x)$ satisfies the so-called Harnack theorem, which tells us that, on any open interval $(a, b) \subset (0, R)$, there is a constant $C_{(a,b)}$ such that

$$\max_{x\in(a,b)} \mid \Psi_0 \mid (x) \leq C_{(a,b)} \min_{x\in(a,b)} \mid \Psi_0 \mid (x) \tag{9}$$

holds true.[18] Now, we take an open interval containing x_0. Harnack theorem implies that $|\Psi_0|(x) \equiv 0$ in this interval. Repeating this argument an appropriate number of times, we find that $\mid \Psi_0 \mid$ must be identically zero in $(0, R)$. This is certainly absurd. Therefore, we reach the conclusion that $\Psi_0(x) = |\Psi_0|(x) > 0$ in $(0, R)$ and is nondegenerate.

The same ideas can be also applied to study the strongly correlated electron systems. First, let us consider a simple model: The antiferromagnetic Heisenberg model on a bipartite lattice. The Hamiltonian of this model is of the following form

$$H_{\mathrm{AF}} = \sum_{<\mathbf{ij}>} J_{\mathbf{ij}} \vec{S}_{\mathbf{i}} \cdot \vec{S}_{\mathbf{j}} = \frac{1}{2} \sum_{<\mathbf{ij}>} J_{\mathbf{ij}} \left(\hat{S}_{\mathbf{i}+} \hat{S}_{\mathbf{j}-} + \hat{S}_{\mathbf{i}-} \hat{S}_{\mathbf{j}+} \right) + \sum_{<\mathbf{ij}>} J_{\mathbf{ij}} \hat{S}_{\mathbf{i}z} \hat{S}_{\mathbf{j}z} \tag{10}$$

with $J_{\mathbf{ij}} \geq 0$. $\vec{S}_{\mathbf{i}}$ is the spin operator at site $\mathbf{i}$. The lattice Λ is assumed to be bipartite and connected by these coupling constants.

Since Hamiltonian (10) commutes with $\hat{S}_z = \sum_{\mathbf{i}} \hat{S}_{\mathbf{i}z}$, S_z is a good quantum number. A natural basis of vectors in subspace $V(S_z = M)$ is given by

$$\chi_\alpha(M) = \mid m_1, m_2, \cdots, m_{N_\Lambda} \rangle \tag{11}$$

where $m_{\mathbf{i}}$ represents the eigenvalue of $\hat{S}_{\mathbf{i}z}$ at site $\mathbf{i}$ and they are subject to the condition $m_1 + m_2 + \cdots + m_{N_\Lambda} = M$. In terms of this basis, the ground state wave function $\Psi_0(M)$ of H_{AF} can be written as

$$\Psi_0(M) = \sum_\alpha C_\alpha \chi_\alpha(M) \tag{12}$$

The sum is over all the possible configurations $\{\chi_\alpha(M)\}$.

However, due to the positive sign of the coupling constants $\{J_{\mathbf{ij}}\}$, it is difficult to uncover directly the sign rule satisfied by $\{C_\alpha\}$. To remedy this problem, one needs to introduce a unitary transformation $\hat{U}_1 = \exp\left(i\pi \sum_{\mathbf{i}\in B} \hat{S}_{\mathbf{i}z}\right)$, which rotates each spin in sublattice B by an angle π about its z-axis.[19] Under this transformation, H_{AF} is mapped onto

$$\widetilde{H}_{\mathrm{AF}} = \hat{U}_1^\dagger H_{\mathrm{AF}} \hat{U}_1 = \sum_{<\mathbf{ij}>} (-J_{\mathbf{ij}}/2) \left(\hat{S}_{\mathbf{i}+} \hat{S}_{\mathbf{j}-} + \hat{S}_{\mathbf{i}-} \hat{S}_{\mathbf{j}+} \right) + \sum_{<\mathbf{ij}>} J_{\mathbf{ij}} \hat{S}_{\mathbf{i}z} \hat{S}_{\mathbf{j}z} \tag{13}$$

Notice that the coupling constants in the spin-flipping terms of $\widetilde{H}_{\mathrm{AF}}$ have negative signs.

For $\widetilde{H}_{\mathrm{AF}}$, we are able to show that the expansion coefficients $\{\widetilde{C}_\alpha\}$ of its ground state $\widetilde{\Psi}_0$ satisfy the Marshall's rule $\widetilde{C}_\alpha > 0$. First, by following the aforementioned

steps, we re-write the ground state energy E_0 as an expectation value of $\widetilde{H}_{\rm AF}$ in its ground state $\widetilde{\Psi}_0$ and observe that, for any pair of indices α and α',

$$\left\langle \chi_\alpha(M) \left| \left(\hat{S}_{\mathbf{i}+}\hat{S}_{\mathbf{j}-} + \hat{S}_{\mathbf{i}-}\hat{S}_{\mathbf{j}+} \right) \right| \chi_{\alpha'}(M) \right\rangle \geq 0 \tag{14}$$

holds true. Consequently, the state $|\widetilde{\Psi}_0|$, which is constructed by replacing each $\widetilde{C}_\alpha$ with $|\widetilde{C}_\alpha|$ in Eq. (12), has a lower energy than $\widetilde{\Psi}_0$. Therefore, it is also a ground state.

Next, we consider the Schrödinger equation of $|\widetilde{\Psi}_0|$. By inequality (14), it can be shown that, if $C_\alpha = 0$ for some index α, then any vector $\chi_{\alpha'}$, which is related to χ_α by a spin-flipping exchange, must have a zero coefficient in the expansion of $|\widetilde{\Psi}_0|$. On the other hand, since the lattice is connected by the coupling constants $\{J_{\mathbf{ij}}\}$, any vector χ_β can be reached from χ_α by a finite number of spin-flipping exchanges. Therefore, by repeating the above process an appropriate number of times, we reach the conclusion that all the expansion coefficients are zero. That is impossible.

As usual, the Marshall's rule satisfied by $\{\widetilde{C}_\alpha\}$ implies that the ground state $\widetilde{\Psi}_0(M)$ of $\widetilde{H}_{\rm AF}$ is nondegenerate in subspace $V(M)$. On the other hand, since $\widetilde{H}_{\rm AF}$ is unitarily equivalent to the original antiferromagnetic Heisenberg Hamiltonian, we conclude that the ground state $\Psi_0(M)$ of the antiferromagnetic Heisenberg Hamiltonian $H_{\rm AF}$ in $V(M)$ is also nondegenerate.

Finally, we consider the Hubbard model. As explained above, we first introduce a unitary transformation which maps the original positive-U Hubbard model into a negative-U Hamiltonian. This can be achieved by the so-called partial particle-hole transformation $\hat{U}_2$,[4] which is defined by

$$\hat{U}_2^\dagger \hat{c}_{\mathbf{i}\uparrow}\hat{U}_2 = \epsilon(\mathbf{i})\hat{c}_{\mathbf{i}\uparrow}^\dagger, \quad \hat{U}_2^\dagger \hat{c}_{\mathbf{i}\downarrow}\hat{U}_2 = \hat{c}_{\mathbf{i}\downarrow}. \tag{15}$$

Under this transformation, the positive-U Hubbard Hamiltonian at half-filling (with $\mu = 0$) is mapped to

$$H_H(-U) = -t\sum_\sigma \sum_{<\mathbf{ij}>} \left(\hat{c}_{\mathbf{i}\sigma}^\dagger \hat{c}_{\mathbf{j}\sigma} + \hat{c}_{\mathbf{j}\sigma}^\dagger \hat{c}_{\mathbf{i}\sigma} \right) - U\sum_{\mathbf{i}\in\Lambda} \left(\hat{n}_{\mathbf{i}\uparrow} - \frac{1}{2} \right) \left(\hat{n}_{\mathbf{i}\downarrow} - \frac{1}{2} \right). \tag{16}$$

When N, the number of electrons in the system, equals an even integer $2M$, the ground state wave function $\widetilde{\Psi}_0(2M)$ of $H_H(-U)$ can be written as

$$\widetilde{\Psi}_0(2M) = \sum_{\alpha,\,\beta} W_{\alpha\beta}\psi_\alpha^\uparrow \otimes \psi_\beta^\downarrow \tag{17}$$

where $\{\psi_\alpha^\sigma\}$ are configurations of electrons of spin σ defined by

$$\psi_\alpha^\sigma = \hat{c}_{\mathbf{i}_1\sigma}^\dagger \hat{c}_{\mathbf{i}_2\sigma}^\dagger \cdots \hat{c}_{\mathbf{i}_M\sigma}^\dagger \mid 0\rangle \tag{18}$$

In Eq. (18), $(\mathbf{i}_1, \mathbf{i}_2, \cdots, \mathbf{i}_M)$ denotes the lattice sites occupied by fermions of spin σ. $\mid 0\rangle$ is the vacuum state. The total number of these configurations is $C_{N_\Lambda}^M$.

In the Hubbard model, electrons are itinerant. Therefore, we have the so-called fermion sign problem. Consequently, the coefficients $\{W_{\alpha\beta}\}$ do not satisfy the simple

Marshall's rule. However, Lieb proved that, if one takes index α for row index and β for column index and writes the coefficients $\{W_{\alpha\beta}\}$ into a matrix $\mathcal{W}$ (with $\mathrm{Tr}\mathcal{W}^{\dagger}\mathcal{W} = 1$), then *this matrix is, in fact, a positive-definite matrix.* It is the generalized Marshall's rule for the ground states of the negative-U Hubbard model.

To prove this fact, we follow the above procedure and rewrite $E_0(-U)$ as the expectation of $H_H(-U)$ in $\widetilde{\Psi}_0(2M)$. A little algebra yields

$$E_0(-U) = 2\mathrm{Tr}(T\mathcal{W}\mathcal{W}) - U\sum_{\mathrm{i}\in\Lambda}\mathrm{Tr}(\mathcal{W}^{\dagger}N_{\mathrm{i}}\mathcal{W}N_{\mathrm{i}}) \tag{19}$$

where T is the matrix of the hopping term of $H_H(-U)$ and N_{i} is the matrix of operator $\hat{n}_{\mathrm{i}} - 1/2$. Since matrix $\mathcal{W}$ is Hermitian, we can find a unitary matrix V, which diagonalizes it. Let $\{w_m\}$ be the eigenvalues of $\mathcal{W}$ and $\{\mid m\rangle\}$ be the column vectors of the diagonalizing matrix V. Then, $E_0(-U)$ can be further reduced to

$$E_0(-U) = 2\sum_{m}\langle m \mid \hat{T} \mid m\rangle w_m^2 - U\sum_{\mathrm{i}\in\Lambda}\sum_{m,\,n} w_m w_n |\langle n \mid \hat{N}_{\mathrm{i}} \mid m\rangle|^2 \tag{20}$$

Obviously, if we replace $\{w_m\}$ with their absolute values, the summations on the right hand side of Eq. (20) becomes less. In other words, by replacing the coefficient matrix $\mathcal{W}$ of $\widetilde{\Psi}_0(2M)$ with $\mid \mathcal{W} \mid$, we obtain a new wave function $|\widetilde{\Psi}_0|(2M)$, which has a lower energy than the ground state $\widetilde{\Psi}_0(2M)$. Therefore, it must be also a ground state of $H_H(-U)$. Furthermore, its coefficient matrix $\mid \mathcal{W} \mid$ is a semipositive definite matrix.

By substituting $|\widetilde{\Psi}_0|(2M)$ into the Schrödinger equation $H_H(-U)|\widetilde{\Psi}_0| = E_0|\widetilde{\Psi}_0|$ and noticing that the lattice Λ is connected by electron hopping, we can further show that, if one eigenvalue $w_m = 0$, then all the eigenvalues of $\mathcal{W}$ are equal to zero. This is impossible. Therefore, $\mathcal{W}$ is actually a positive definite matrix and the ground state $\widetilde{\Psi}_0(2M)$ is nondegenerate. Since $H_H(-U)$ is unitarily equivalent to $H_H(U)$ at half-filling, the ground state of the latter Hamiltonian is also nondegenerate.

With the positive definiteness of the coefficient matrix $\mathcal{W}$ of $\widetilde{\Psi}_0(2M)$, inequality (4) can be proven as a direct corollary. Similarly, other results listed above are also based on this fact, although their proofs require a little more effort.

In summary, the spin-reflection-positivity method reveals that the ground states of some strongly correlated electron systems satisfy the Marshall's rule in a more sophisticated manner. In return, this rule enables us to establish several important qualitative properties on the spin and superconducting correlations as well as the excitation gaps in these systems.

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AN ALGEBRAIC APPROACH TO THE EIGENSTATES OF THE CALOGERO MODEL

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An algebraic treatment of the eigenstates of the (A_{N-1}-)Calogero model is presented, which provides an algebraic construction of the nonsymmetric orthogonal eigenvectors, symmetrization, antisymmetrization and calculation of square norms in a unified way.

1. The Calogero Model

In 1990's, one-dimensional quantum integrable systems with inverse-square long-range interactions[1–3] attracted renewed interests of mathematicians and physicists since their relationships with the theory of multivariable orthogonal polynomials[4,5] are recognized. The (A_{N-1}-)Calogero model[1] with distinguishable particles[6]

$$\hat{\mathcal{H}}^{(A)} = \frac{1}{2}\sum_{j=1}^{N}\Big(-\frac{\partial^2}{\partial x_j^2} + \omega^2 x_j^2\Big) + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\frac{a^2 - aK_{jk}}{(x_j - x_k)^2}, \quad \omega \in \mathbb{R}_{>0}, \quad a \in \mathbb{R}_{\geq 0}, \tag{1}$$

where $(K_{jk}f)(\cdots, x_j, \cdots, x_k, \cdots) = f(\cdots, x_k, \cdots, x_j, \cdots)$, $j, k \in \{1, 2, \cdots, N\}$, is known to have the nonsymmetric multivariable Hermite polynomial as the polynomial part of the joint eigenvector of the conserved operators.[7–9] We introduce a transformed Hamiltonian whose eigenvectors are polynomials,

$$\mathcal{H}^{(A)} := \big(\phi_{\mathrm{g}}^{(A)}(x)\big)^{-1} \circ \big(\hat{\mathcal{H}}^{(A)} - E_{\mathrm{g}}^{(A)}\big) \circ \phi_{\mathrm{g}}^{(A)}(x), \tag{2}$$

where the reference state and its eigenvalue are

$$\phi_{\mathrm{g}}^{(A)}(x) = \prod_{1\leq j<k\leq N} |x_j - x_k|^a \exp\Big(-\frac{1}{2}\omega\sum_{m=1}^{N} x_m^2\Big),$$
$$E_{\mathrm{g}}^{(A)} = \frac{1}{2}\omega N\big(Na + (1-a)\big).$$

We shall deal with the eigenvectors of the Hamiltonian (2) in $\mathbb{C}[x]$, the polynomial ring with N variables over $\mathbb{C}$, while the eigenstates for the original Hamiltonian (1)

is in $\mathbb{C}[x]\phi_{\mathrm{g}}^{(A)} := \{f(x)\phi_{\mathrm{g}}^{(A)}(x) | f \in \mathbb{C}[x]\}$. The Hamiltonian (2) is Hermitian with respect to the inner product on $\mathbb{C}[x]$,

$$\langle f, g\rangle_{(A)} := \int_{-\infty}^{\infty} \prod_{j=1}^{N} \mathrm{d}x_j |\phi_{\mathrm{g}}^{(A)}(x)|^2 \overline{f(x)} g(x), \quad f, g \in \mathbb{C}[x], \tag{3}$$

where $\overline{f(x)}$ denotes the complex conjugate of $f(x)$. The inner product is induced from the natural inner product on $\mathbb{C}[x]\phi_{\mathrm{g}}^{(A)}$. The reference state corresponds to the weight function in the inner product $\langle \cdot, \cdot \rangle_{(A)}$.

The commutative conserved operators for the Hamiltonian are known to be the Cherednik operators.[10] To show this, we have to introduce the Dunkl operators,[11]

$$\nabla_j^{(A)} := \frac{\partial}{\partial x_j} + a \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{1}{x_j - x_k}(1 - K_{jk}),$$

and the creation-like and annihilation-like operators,

$$\alpha_l^{(A)\dagger} := x_l - \frac{1}{2\omega}\nabla_l^{(A)}, \quad \alpha_l^{(A)} = \frac{1}{2\omega}\nabla_l^{(A)},$$

in $\in \mathrm{End}(\mathbb{C}[x])$, where the superscript $\dagger$ on any operator denotes its Hermitian conjugate with respect to the inner product (3). From these operators, a set of Hermitian and commutative differential operators, $d_j^{(A)} \in \mathrm{End}(\mathbb{C}[x])$, $[d_j^{(A)}, d_k^{(A)}] = 0$, is constructed by

$$d_j^{(A)} := 2\omega \alpha_j^{(A)\dagger} \alpha_j^{(A)} + a \sum_{k=j+1}^{N} K_{jk},$$

which we call the Cherednik operators.[10,12] The Hamiltonian (2) can be expressed as

$$\mathcal{H}^{(A)} = \omega \sum_{l=1}^{N} \Big(d_l^{(A)} - \frac{1}{2} a(N-1) \Big).$$

Thus we conclude that the Cherednik operators $\{d_j^{(A)} | j = 1, 2, \cdots, N\}$ give a set of commutative conserved operators of the Calogero model.

2. The Nonsymmetric Multivariable Hermite Polynomial

The Cherednik operators define inhomogeneous multivariable polynomials as their joint polynomial eigenvectors, which are nothing but the nonsymmetric multivariable Hermite polynomials that form an orthogonal basis of the polynomial ring $\mathbb{C}[x]$.[7–9,13] Let $\check{I} := \{1, 2, \cdots, N-1\}$ and $I := \{1, 2, \cdots, N\}$ be sets of indices and let V be an N-dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$. We take an orthonormal basis of V $\{\varepsilon_j | j \in I\}$ such that $\langle \varepsilon_j, \varepsilon_k \rangle = \delta_{jk}$. The A_{N-1}-type root system R associated with the simple Lie algebra of type A_{N-1} is realized as $R = \{\varepsilon_j - \varepsilon_k | j, k \in I, j \neq k\} (\subset V)$. A root basis of R is defined by

$\Pi := \{\alpha_j = \varepsilon_j - \varepsilon_{j+1} | j \in \check{I}\}$, whose elements are called simple roots. Let R_+ be positive roots relative to Π and $R_- := -R_+$. The root lattice Q is defined by $Q := \bigoplus_{j\in\check{I}} \mathbb{Z}\alpha_j$ and positive root lattice Q_+ is defined by replacing $\mathbb{Z}$ with $\mathbb{Z}_{\geq 0}$.

A reflection on V with respect to the hyperplane that is orthogonal to a root α is expressed by $s_\alpha(\mu) := \mu - \langle \alpha^\vee, \mu\rangle \alpha$, where $\alpha^\vee := \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ is a coroot corresponding to α. The A_{N-1}-type Weyl group is generated by $\{s_j := s_{\alpha_j} | \alpha_j \in \Pi\}$ which is isomorphic to $\mathfrak{S}_N$. For each $w \in W$, we define $R_w := R_+ \cap w^{-1}R_-$. We denote by $\ell(w)$ the length of $w \in W$ defined by $\ell(w) := |R_w|$. When $w \in W$ is expressed as a product of simple reflections, $w = s_{j_k} \cdots s_{j_2} s_{j_1}$, with $k = \ell(w)$, we call it reduced. By use of the reduced expression, the set R_w is given by $R_w = \{\alpha_{j_1}, s_{j_1}(\alpha_{j_2}), \cdots, s_{j_1} s_{j_2} \cdots s_{j_{l-1}}(\alpha_{j_l})\}$.

We introduce lattices $P := \bigoplus_{j\in I} \mathbb{Z}_{\geq 0}\varepsilon_j$ and $P_+ := \{\mu = \sum_{j\in I} \mu_j \varepsilon_j \in P | \mu_1 \geq \mu_2 \geq \cdots \geq \mu_N \geq 0\}$, whose elements are called a composition and a partition, respectively. The lattice P is W-stable. The degree of the composition and partition is denoted by $|\mu| := \sum_{j\in I} \mu_j$. Let $W(\mu) := \{w(\mu) | w \in W\}$ be the W-orbit of $\mu \in P$. In a W-orbit of $W(\mu)$, there exists a unique partition $\mu^+ \in P_+$ such that $\mu = w(\mu^+) \in P$ $(w \in W)$. We define $\rho := \frac{1}{2}\sum_{\alpha\in R_+} \alpha = \frac{1}{2}\sum_{j\in I}(N - 2j + 1)\varepsilon_j$ and $1^N := \sum_{j\in I} \varepsilon_j$. We introduce the following operator $d^{(A)\lambda} := \sum_{j\in I} \lambda_j d_j^{(A)}$, $\lambda \in P$, $j \in \check{I}$, which relates the Cherednik operators with lattice P, so that we can deal with the eigenvalues of the Cherednik operators in terms of the lattice P.

We identify the elements of the lattice P with those of the polynomial ring over $\mathbb{C}$, $x^\mu := x_1^{\mu_1} x_2^{\mu_2} \cdots x_N^{\mu_N} \in \mathbb{C}[x]$. We denote the shortest element of W such that $w_\mu^{-1}(\mu) \in P_+$ by w_μ and define $\rho(\mu) := w_\mu(\rho)$. We introduce an order $\preceq$ on P,

$$\nu \preceq \mu \quad (\nu, \mu \in P) \Leftrightarrow \begin{cases} \nu^+ \overset{\mathrm{d}}{<} \mu^+ & \nu \notin W(\mu^+) \\ \mu - \nu \in Q_+ & \nu \in W(\mu^+), \end{cases}$$

where the symbol $\overset{\mathrm{d}}{<}$ denotes the dominance order among partitions,

$$\nu \overset{\mathrm{d}}{<} \mu \quad (\mu, \nu \in P_+) \Leftrightarrow \mu \neq \lambda,\ |\mu| = |\nu| \text{ and } \sum_{k=1}^{l} \nu_k \leq \sum_{k=1}^{l} \mu_k,$$

for all $l \in I$. The definition of the nonsymmetric multivariable Hermite polynomial is summarized as follows.

Definition 1: The (monic) nonsymmetric multivariable Hermite polynomial $h_\mu^{(A)} \in \mathbb{C}[x]$, $\mu \in P$, as the joint eigenvector for the commutative Cherednik operators $\{d^{(A)\lambda} | \lambda \in P\}$, is presented by

$$h_\mu^{(A)}(x) = x^\mu + \sum_{\substack{\nu \preceq \mu \\ \text{or } |\nu|<|\mu|}} v_{\mu\nu}^{(A)}(a, \frac{1}{2\omega}) x^\nu \in \mathbb{C}[x],$$

$$d^{(A)\lambda} h_\mu^{(A)} = \langle \lambda, \mu + a\rho(\mu) + \frac{1}{2}a(N-1)1^N \rangle h_\mu^{(A)}. \tag{4}$$

Since $d^{(A)\lambda}$ is a Hermitian operator with respect to the inner product (3) and all the simultaneous eigenspaces of the Cherednik operators $\{d^{(A)\lambda}\}$ are one-dimensional in the sense that the eigenvalues of $\{d^{(A)\lambda}\}$ are non-degenerate, it proves that the polynomials $h_\mu^{(A)}$ are orthogonal with respect to the inner product, i.e., $\langle h_\mu^{(A)}, h_\nu^{(A)}\rangle_{(A)} = \delta_{\mu,\nu}||h_\mu^{(A)}||^2$. Actually, the nonsymmetric multivariable Hermite polynomials form an orthogonal basis in $\mathbb{C}[x]$.

3. The Rodrigues Formula

Here we show the Rodrigues formula that generates the monic nonsymmetric multivariable Hermite polynomial. We introduce the Knop-Sahi operators $\{e^{(A)}, e^{(A)\dagger}\}$[7,14] and the braid operators $\{S_j^{(A)} | j \in \check{I}\}$ defined by

$$e^{(A)} := \alpha_1^{(A,B)} K_1 K_2 \cdots K_{N-1},\ e^{(A)\dagger} = K_{N-1} \cdots K_2 K_1 \alpha_1^{(A,B)\dagger},\ S_j^{(A)} := [K_j, d_j^{(A)}],$$

where $K_j := K_{j,j+1}$, $j \in \check{I}$. The operators $\{S_j^{(A)}, e^{(A)\dagger}\}$ intertwine the simultaneous eigenspaces of $\{d^{(A)\lambda}\}$. The raising operators $\{A_\mu^{(A)\dagger} | \mu \in P_+\}$ are defined by

$$A_\mu^{(A)\dagger} := (A_1^{(A,B)\dagger})^{\mu_1-\mu_2} (A_2^{(A,B)\dagger})^{\mu_2-\mu_3} \cdots (A_N^{(A,B)\dagger})^{\mu_N},$$
$$A_j^{(A)\dagger} := (S_j^{(A,B)} S_{j+1}^{(A,B)} \cdots S_{N-1}^{(A,B)} e^{(A,B)\dagger})^j, \quad j \in I.$$

Let S_{w_μ} be defined by $S_{w_\mu} := S_{j_l} \cdots S_{j_2} S_{j_1}$, where $w_\mu = s_{j_l} \cdots s_{j_2} s_{j_1}$ is one of the reduced expressions of w_μ. Then we can show the following relations,

$$d^{(A)\lambda} A_\mu^{(A)\dagger} = A_\mu^{(A)\dagger} (d^{(A)\lambda} + \langle \lambda, \mu \rangle), \quad \left[A_\mu^{(A)\dagger}, A_\nu^{(A)\dagger}\right] = 0, \quad \text{for } \mu, \nu \in P_+,$$
$$S_{w_\mu}^{(A)} d^{(A)\lambda} = d^{(A)s_{w_\mu}(\lambda)} S_{w_\mu}^{(A}, \quad \text{for } \mu \in P, \tag{5}$$

which lead to the Rodrigues formula for the monic nonsymmetric Hermite polynomial.[9]

Theorem 2: The monic nonsymmetric multivariable Hermite polynomial $h_\mu^{(A)}$ with a general composition $\mu \in P$ in the W-orbit of the partition $\mu^+ \in P_+$ is algebraically obtained by applying the raising operator $A_{\mu^+}^{(A)}$ and the product of braid operators $S_{w_\mu}^{(A)}$ to $h_0^{(A)} = 1$,

$$h_\mu^{(A)} = (c_{w_\mu}^{(A)} c_{\mu^+}^{(A)})^{-1} S_{w_\mu}^{(A)} A_{\mu^+}^{(A)\dagger} h_0^{(A)},\ h_0^{(A,B)} = 1,\ \mu \in W(\mu^+),\ \mu^+ \in P_+,$$

where the coefficient of the top term is expressed as

$$c_{\mu^+}^{(A)} := \prod_{\alpha \in R_+} \prod_{l=1}^{\langle \alpha^\vee, \mu \rangle} \left(-l - a\langle \alpha^\vee, \rho \rangle\right), \quad c_{w_\mu}^{(A)} := \prod_{\alpha \in R_{w_\mu}} \frac{\langle \alpha^\vee, \mu^+ + a\rho \rangle^2 - a^2}{\langle \alpha^\vee, \mu^+ + a\rho \rangle}.$$

Proof: Using Eq. (5), we can confirm that $h_\mu^{(A)}$ given by the above formula satisfies definition 1. □

Using the square norm for $h_0^{(A)} = 1$,

$$\langle h_0^{(A)}, h_0^{(A)} \rangle_{(A)} = \frac{(2\pi)^{\frac{N}{2}}}{(2\omega)^{\frac{1}{2}N(Na+(1-a))}} \prod_{j\in I} \frac{\Gamma(1+ja)}{\Gamma(1+a)},$$

which is proved by a certain limit of the Selberg integral,[15] and the Rodrigues formula, we can calculate the square norm of the nonsymmetric multivariable Hermite polynomial in an algebraic fashion.[9]

Theorem 3: The square norms of the nonsymmetric multivariable Hermite polynomial $h_\nu^{(A)}$ with a general composition $\mu \in W(\mu^+)$, $\mu^+ \in P_+$ is given by[9]

$$\begin{aligned}
&\langle h_\mu^{(A)}, h_\nu^{(A)} \rangle_{(A)} \\
&= \delta_{\mu,\nu} \frac{(2\pi)^{\frac{N}{2}}}{(2\omega)^{\frac{1}{2}N(Na+(1-a))+|\mu|}} \prod_{\beta\in R_{w_\mu}} \frac{\langle \beta^\vee, \mu^+ + a\rho\rangle^2}{\langle \beta^\vee, \mu^+ + a\rho\rangle^2 - a^2} \prod_{i\in I} \Gamma\big(\mu_i^+ + a(N-i)+1\big) \\
&\quad \prod_{\alpha\in R_+} \frac{\Gamma\big(\langle \alpha^\vee, \mu^+ + a\rho\rangle + 1 + a\big)\Gamma\big(\langle \alpha^\vee, \mu^+ + a\rho\rangle + 1 - a\big)}{\Gamma\big(\langle \alpha^\vee, \mu^+ + a\rho\rangle + 1\big)^2}.
\end{aligned}$$

Proof: The above formula can be verified by use of relations among raising and braid operators and evaluation of the eigenvalues of the Cherednik operators. □

4. Symmetrization and Antisymmetrization

The nonsymmetric multivariable Hermite polynomials with compositions μ in the same W-orbit of the partition μ^+ share the same eigenvalue of the Hamiltonian (2),

$$\mathcal{H}^{(A)} h_\mu^{(A)} = \omega|\mu^+|h_\mu^{(A)}, \quad \text{for } \mu \in W(\mu^+), \quad \mu^+ \in P_+.$$

More generally, the polynomial with compositions $\mu \in W(\mu^+)$ share the same eigenvalue of an arbitrary symmetric polynomial, e.g., any of the power sums, of the Cherednik operators. Thus any linear combinations of $h_\mu^{(A)}$, $\mu \in W(\mu^+)$, $\mu^+ \in P_+$ are joint eigenvectors of the Calogero Hamiltonian (2) and its higher-order conserved operators.

Among all such linear combinations, we consider eigenvectors of the Calogero Hamiltonian (2) in W-symmetric and W-antisymmetric polynomial rings over $\mathbb{C}$, $\mathbb{C}[x]^{\pm W}$. In our formulation, we do not use symmetrizer or anti-symmetrizer[7] which makes the coefficients of the top terms differ from unity. We introduce a sublattice of P_+ by $P_+ + \delta := \{\mu + \delta | \mu \in P_+\}$, $\delta := \sum_{j\in I}(N-j)\varepsilon_j$ to describe the antisymmetric eigenvectors.

Theorem 4: [9] Let $H_{\mu^+}^{(A)+}$ for $\mu^+ \in P_+$ and $H_{\mu^+}^{(A)-}$ for $\mu^+ \in P_+ + \delta$ be the following linear combination of the nonsymmetric multivariable Hermite polynomials with compositions $\mu \in W(\mu^+)$,

$$H_{\mu^+}^{(A)\pm} = \sum_{\mu\in W(\mu^+)} b_{\mu^+\mu}^{(A)\pm} h_\mu^{(A)}, \tag{6}$$

whose coefficients are

$$b^{(A)\pm}_{\mu^+\mu} = \prod_{\alpha\in R_{w_\mu}} \pm\frac{\langle\alpha,\mu^+ + a\rho\rangle \mp a}{\langle\alpha,\mu^+ + a\rho\rangle}. \tag{7}$$

Then the polynomials are in symmetric or antisymmetric polynomial rings over $\mathbb{C}$, i.e., $H^{(A)\pm}_{\mu^+} \in \mathbb{C}[x]^{\pm W}$. We call them the symmetric and antisymmetric multivariable Hermite polynomials, respectively.

Proof: Requiring the linear combination of the nonsymmetric polynomial (6) to satisfy $K_j H^{(A)\pm}_\mu = \pm H^{(A)\pm}_\mu$, $b^{(A)\pm}_{\mu^+\mu^+} = 1$, we obtain the coefficients $b^{(A)\pm}_{\mu^+\mu}$ as shown in Eq.(7). □

The symmetric and antisymmetric multivariable Hermite polynomials are identified by the polynomial parts of the eigenstates for all the conserved operators of the (A_{N-1}-)Calogero model with indistinguishable (bosonic or fermionic) particles,[1,17,18]

$$\hat{\mathcal{H}}^{(A)\pm}(a) = \frac{1}{2}\sum_{j=1}^{N}\Big(-\frac{\partial^2}{\partial x_j^2} + \omega^2 x_j^2\Big) + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\frac{a^2 \mp a}{(x_j - x_k)^2},$$

which is obtained by restricting the operand of the Hamiltonian (1) to the space $\mathbb{C}[x]^{\pm W}\phi_{\mathrm{g}}^{(A)}$, $\hat{\mathcal{H}}^{(A)}\big|_{\mathbb{C}[x]^{\pm W}\phi_{\mathrm{g}}^{(A)}} = \hat{\mathcal{H}}^{(A)\pm}(a)$.

From the square norms of the nonsymmetric polynomials $\langle h^{(A)}_\mu, h^{(A)}_\mu\rangle_{(A)}$ and the coefficients $b^{(A)\pm}_{\mu^+\mu}$, we can evaluate the square norms of the (anti-)symmetric multivariable Hermite polynomials. To prove the formula of the square norms, we need the following lemma,[16]

Lemma 5: *For $\mu \in P_+$, we have an identity,*

$$\sum_{\nu\in W(\mu)}\prod_{\alpha\in R_{w_\nu}}\frac{\langle\alpha^\vee,\mu + a\rho\rangle \mp a}{\langle\alpha^\vee,\mu + a\rho\rangle \pm a} = N!\prod_{\alpha\in R_+}\frac{\langle\alpha^\vee,\mu + a\rho\rangle}{\langle\alpha^\vee,\mu + a\rho\rangle \pm a}.$$

The lemma is proved by use of an expression of the Poincaré polynomials.[19]

Theorem 6: [9] Let $H^{(A)+}_{\mu^+}$ for $\mu^+ \in P_+$ and $H^{(A)-}_{\mu^+}$ for $\mu^+ \in P_+ + \delta$. The square norms of the (anti-)symmetric multivariable Hermite polynomials are given by

$$\langle H^{(A)\pm}_\mu, H^{(A)\pm}_\nu\rangle_{(A)} = \delta_{\mu,\nu}\frac{(2\pi)^{\frac{N}{2}}N!}{(2\omega)^{\frac{1}{2}N(Na+(1-a))+|\mu|}}\prod_{j\in I}\Gamma\big(\mu_j + a(N-j)+1\big)$$
$$\prod_{\alpha\in R_+}\frac{\Gamma\big(\langle\alpha^\vee,\mu+a\rho\rangle + 1 \mp a\big)\Gamma\big(\langle\alpha^\vee,\mu+a\rho\rangle \pm a\big)}{\Gamma\big(\langle\alpha^\vee,\mu+a\rho\rangle + 1\big)\Gamma\big(\langle\alpha^\vee,\mu+a\rho\rangle\big)}.$$

Proof: The proof is straightforward from theorems 3 and 4, and lemma 5. □

5. Concluding Remarks

We have presented the Rodrigues formula for the monic nonsymmetric multivariable Hermite polynomial which gives the nonsymmetric orthogonal eigenfunctions of the (A_{N-1}-)Calogero model with distinguishable particles. Through (anti-)symmetrization, we have constructed the (anti-)symmetric Hermite polynomials that give the polynomial parts of the eigenfunctions of the (A_{N-1}-)Calogero model with distinguishable particles. The square norms of the above three cases are calculated in an algebraic manner. Our formulation in this work is also applicable to the B_N-Calogero models with distinguishable or indistinguishable particles and results in the Rodrigues formula for the monic nonsymmetric multivariable Laguerre polynomial and square norms of the nonsymmetric and (anti-)symmetric multivariable Laguerre polynomials.[9]

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ONSAGER'S ALGEBRA AND PARTIALLY ORTHOGONAL POLYNOMIALS

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The energy eigenvalues of the superintegrable chiral Potts model are determined by the zeros of special polynomials which define finite representations of Onsager's algebra. The polynomials determining the low-sector eigenvalues have been given by Baxter in 1988. In the $\mathbb{Z}_3$−case they satisfy 4-term recursion relations and so cannot form orthogonal sequences. However, we show that they are closely related to Jacobi polynomials and satisfy a special "partial orthogonality" with respect to a Jacobi weight function.

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1. Introduction

F.Y.Wu and Y.K.Wang[1] were the first to consider the Potts model with chiral interaction terms. Their interest in this generalization arose from duality considerations, but the idea proved to be very fruitful in many respects: Ostlund[2] and Huse[3] proposed the chiral Potts (CP) model for phenomenological applications: it allows to describe incommensurate phases using nearest neighbor interactions only. We give a few references[4–7] from which the subsequent development can be traced, and turn directly to the superintegrable chiral $\mathbb{Z}_N$ Potts quantum chain.[8] This is a particularly interesting model, because it provides some of the rare representations known for Onsager's algebra[9] and in this sense generalizes the Ising quantum chain (for $N = 2$ it is the Ising model). Integrability by Onsager's algebra entails that all eigenvalues of the hamiltonian are determined by the zeros of certain polynomials, which for the chiral Potts model were first derived by Baxter.[10] Although the definition of Baxter's polynomials looks very simple, the properties of these polynomials turn out to be quite non-trivial and interesting.[11] The main part of this note deals with the properties of these polynomials. They satisfy $N + 1$-term recursion relations, therefore for $N > 2$ they cannot form orthogonal sequences. However, as found recently,[11] several properties which characterize orthogonal polynomials

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are almost true for Baxter's polynomials (e.g. the zero separation property is true except for one extreme zero).

We first recall the definitions of the superintegrable CP-hamiltonian and Onsager's algebra and then, following B.Davies,[12] we sketch how the formula for the energy eigenvalues emerges. We consider Baxter's polynomials and their recursion relations. Equivalent polynomials with their zeros in $(-1,+1)$ for $N=3$ are written in terms of a determinant. Their expansion in terms of Jacobi polynomials gives the surprising result that many of the expansion coefficients vanish, leading to the notion of "partial orthogonality".

The hamiltonian defining the $\mathbb{Z}_N$-superintegrable chiral Potts quantum chain[6,8] is:

$$\mathcal{H} \;=\; -\sum_{j=1}^{L}\sum_{l=1}^{N-1} \frac{2}{1-\omega^{-l}}\left(X_j^l \;+\; k\, Z_j^l Z_{j+1}^{N-l}\right). \tag{1}$$

Here $\omega = e^{2\pi i/N}$ and Z_j and X_j are $\mathbb{Z}_N$-spin operators acting in the vector spaces $\mathbb{C}^N$ at the sites $j=1,2,\ldots,L$ (L is the chain length). The operators obey $Z_i X_j = X_j Z_i \omega^{\delta_{i,j}}$; $Z_j^N = X_j^N = 1$ and we assume $X_{L+1} = X_1$ (periodic b.c). A convenient representation is $(X_j)_{l,m} = \delta_{l,m+1} \mod N$ and $(Z_j)_{l,m} = \delta_{l,m}\omega^m$. For $N \geq 3$ the complex coefficients make the chain hamiltonian parity non-invariant. For $N=2$ we get the Ising quantum chain. For fixed N there is only one parameter, the temperature variable k. Incommensurate phases arise due to ground state level crossings. $\mathcal{H}$ commutes with the $\mathbb{Z}_N$-charge $\widehat{Q} = \prod_{j=1}^{L} Z_j$. We write the eigenvalues of $\widehat{Q}$ as ω^Q. $Q = 0,1,\ldots,N-1$ labels the charge sectors of $\mathcal{H}$.

We split $\mathcal{H}$ into two operators writing $\mathcal{H}^{(s)} = -\frac{1}{2}N(A_0 + kA_1)$. A remarkable property of $\mathcal{H}$ is that A_0 and A_1 satisfy[8] the Dolan-Grady[13] relations

$$[A_0,[A_0,[A_0,A_1]]] \;=\; 16\,[A_0,A_1]\,; \qquad\qquad [A_1,[A_1,[A_1,A_0]]] \;=\; 16\,[A_1,A_0]\,,$$

which are the conditions[14] for A_0 and A_1 to generate Onsager's algebra $\mathcal{A}$, which is formed[9] from elements A_m, G_l, $m \in \mathbb{Z}$, $l \in \mathbb{N}$, $l \geq m$, satisfying

$$[A_l, A_m] = 4\,G_{l-m}; \qquad [G_l, A_m] = 2\,A_{m+l} - 2\,A_{m-l}; \qquad [G_l, G_m] = 0. \tag{2}$$

From (2), there is a set of commuting operators which includes $\mathcal{H}$:

$$Q_m = \tfrac{1}{2}\left(A_m + A_{-m} + k(A_{m+1} + A_{-m+1})\right); \qquad [\,Q_l,\, Q_m] = 0; \qquad Q_0 = \mathcal{H}.$$

To obtain finite dimensional representations of $\mathcal{A}$ we require the A_m (and analogously the G_l)[12,15] to satisfy a finite difference equation: $\sum_{k=-n}^{n} \alpha_k\, A_{k-l} \;= 0$. This is solved introducing the polynomial (the main object of the present paper):

$$\mathcal{F}(z) = \sum_{k=-n}^{n} \alpha_k\, z^{k+n} \tag{3}$$

(from $\mathcal{A}$ the α_k are either even or odd in k). Now the A_m and G_m can be expressed in terms of the zeros z_j of $\mathcal{F}(z)$ and the set of operators $E_j^{\pm}$, H_j:

$$A_m = 2\sum_{j=1}^{n} \left(z_j^m\, E_j^+ + z_j^{-m} E_j^-\right); \qquad G_m = \sum_{j=1}^{n} \left(z_j^m - z_j^{-m}\right)\, H_j. \tag{4}$$

From $\mathcal{A}$ these operators obey $sl(2,C)$-commutation rules:

$$[E_j^+, E_k^-] = \delta_{jk}\, H_k; \qquad [H_j,\, E_k^{\pm}] = \pm 2\,\delta_{jk}\, E_k^{\pm}.$$

So $\mathcal{A}$ is isomorphic to a subalgebra of the loop algebra of a sum of $sl(2,C)$ algebras. From the first of eqs.(4) we can express $\mathcal{H}$ in terms of the z_j and the operators $E_j^{\pm}$. Writing $E_j^{\pm} = J_{x,j} \pm iJ_{y,j}$, then in a representation $\mathcal{Z}(n,s)$ characterized by the polynomial zeros $z_1, \ldots, z_n$ and a spin-s representation $\vec{J}_j^{(s)}$ of all the $\vec{J}_j$, we get:

$$\begin{aligned}(A_0 + kA_1)_{\mathcal{Z}(n,s)} &= 2\sum_{j=1}^{n} \left\{(2 + k\,(z_j + z_j^{-1}))J_{x,j}^{(s)} + i(z_j - z_j^{-1})\, J_{y,j}^{(s)}\right\} \\ &= 4\sum_{j=1}^{n} \sqrt{1 + 2k\,c_j + k^2}\; J'^{(s)}_{x,j}\end{aligned}$$

where $J'_{x,j}$ is a rotated $SU(2)$-operator, and $c_j \equiv \cos\theta_j = \frac{1}{2}(z_j + z_j^{-1})$.

For the CP-hamiltonians (1) the spin representation turns out to be $s = \frac{1}{2}$. Accordingly, all eigenvalues of (1) have the form

$$E^{(s)} = -N\left(a + bk + 2\textstyle\sum_{j=1}^{n} m_j \sqrt{1 + 2k\cos\theta_j + k^2}\right), \quad m_j = \pm\tfrac{1}{2}. \tag{5}$$

a and b are non-zero if the trace of A_0 and A_1 is non-zero.

2. Baxter's polynomials

No direct way is known to find the polynomials $\mathcal{F}(z)$ from the hamiltonian $\mathcal{H}$. However, the invention of the two-dimensional integrable CP model,[16,17] which contains $\mathcal{H}$ as a special logarithmic derivative, and functional relations for its transfer matrix have enabled Baxter[10] to obtain the polynomials for the simplest sector of $\mathcal{H}$, which at high-temperatures contains the ground state (the polynomials corresponding to all other sectors have been obtained subsequently in[18–20]). Here we shall consider only the simplest case. Baxter[10] finds that in terms of the variable t or $s \equiv t^N = (c-1)/(c+1)$ (recall $c = \cos\theta$ of (5)) these polynomials take the form

$$P_Q^{(L)}(s) = \frac{1}{N}\sum_{j=0}^{N-1} \left(\frac{1 - t^N}{1 - \omega^j\, t}\right)^L (\omega^j\, t)^{-\sigma_{Q,L}}; \quad \sigma_{Q,L} = (N-1)(L+Q) \bmod N. \tag{6}$$

Here Q denotes the $\mathbb{Z}_N$-charge sector. For $\mathbb{Z}_3$ eq.(6) is, written more explicitly:

$$P_Q^{(L)}(s) = \frac{t^{-\sigma_{Q,L}}}{3}\left\{(t^2 + t + 1)^L + \omega^Q(t^2 + \omega^2 t + \omega)^L + \omega^{-Q}(t^2 + \omega t + \omega^2)^L\right\}.$$

Due to their $\mathbb{Z}_N$-invariance $t \to \omega t$, the $P_Q^{(L)}$ depend only on $s = t^N$. The degree of the $P_Q^{(L)}(s)$ in the variable s is $b_{L,Q} = [((N-1)L - Q)/N]$ where $[x]$ denotes the integer part of x. Considering sequences of these polynomials for fixed Q and $L \in \mathbb{N}$, we notice that the dimensions $b_{L,Q}$ do not always increase by one when increasing L by one: at every Nth step the dimension stays the same: e.g. the dimensions of the $P_{Q=0}^{L}$ for L mod $N = 0$ and L mod $N = 1$ coincide, see Table 1.

The polynomials (6) have their zeros all on the negative real s-axis: $\mathcal{H}$ is hermitian and so in (5) we must have $-1 \le c_j \le +1$ which means negative s_j. We will prefer to deal mostly with equivalent polynomials in the variable c, defining

$$\Pi_Q^{(L)}(c) = (c+1)^{b_{L,Q}} P_Q^{(L)}(s = \tfrac{c-1}{c+1}). \tag{7}$$

Our main concern in this paper is to learn about the properties of the $\Pi_Q^{(L)}(c)$ or $P_Q^{(L)}(s)$, e.g. whether these can be arranged into orthogonal sequences etc. We will find that the $\Pi_Q^{(L)}(c)$ are polynomials with quite remarkable properties. A number of special features of the $\Pi_Q^{(L)}$ have been discussed recently.[11] Here we give some more detailed results for the $\mathbb{Z}_3$-case. As the recursion relations for the $\Pi_Q^{(L)}$ contain a lot of information, we now show how to obtain these.

3. Recursion relations

We start with the observation that the coefficients of the $P_Q^{(L)}(s)$ can be obtained from the expansion of $(1+t+t^2+\ldots+t^{N-1})^L$, simply by taking every $N-$th term of the expansion, starting with the coefficient of $t^{(N-1)L-Q}$. More precisely, we claim that we can define the $P_Q^{(L)}$ by the decomposition

$$(1+t+t^2+\ldots+t^{N-1})^L = \left(\frac{1-t^N}{1-t}\right)^L = \sum_{Q=0}^{N-1} t^{\sigma_{L,Q}} P_Q^{(L)}(s) \tag{8}$$

demanding the $P_Q^{(L)}$ to depend on $s = t^N$ only. Proof: Insert (6) into (8) to get:

$$\left(\frac{1-t^N}{1-t}\right)^L = (1-t^N)^L \, \frac{1}{N} \sum_{Q=0}^{N-1} \sum_{j=0}^{N-1} \frac{\omega^{j(L+Q)}}{(1-\omega^j t)^L}.$$

Interchanging the Q- and j-summations we see that the Q-summation gives zero for $j \neq 0$, leaving only the $j = 0$ term, which is (8).
Eq. (8) can now be used to obtain recursion relations: Write

$$(1+t+t^2+\ldots+t^{N-1}) \sum_{Q=0}^{N-1} t^{\sigma_{L,Q}} \, P_Q^{(L)}(s) \;=\; \sum_{Q=0}^{N-1} t^{\sigma_{L+1,Q}} \, P_Q^{(L+1)}(s). \tag{9}$$

Comparing powers of t, e.g. for $L \mod N = 0$ this gives

$$\begin{pmatrix} P_0^{(L+1)} \\ P_1^{(L+1)} \\ P_2^{(L+1)} \\ \vdots \\ P_{N-1}^{(L+1)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & s & 1 & \dots & 1 \\ 1 & s & s & \dots & 1 \\ \vdots & \vdots & & \vdots & \\ 1 & s & s & \dots & s \end{pmatrix} \begin{pmatrix} P_0^{(L)} \\ P_1^{(L)} \\ P_2^{(L)} \\ \vdots \\ P_{N-1}^{(L)} \end{pmatrix}, \tag{10}$$

For $L \mod N = k$ replace $P_Q \to P_{Q-k}$ cyclically in both column vectors, keeping the same square matrix. Recursion relations not coupling $P_Q^{(L)}$ with different Q follow by the $N-$fold application of these relations, leading to $N+1$-term recursion formulae. These can be transcribed into the corresponding formulae for the $\Pi_Q^{(L)}$. For the Ising case $\mathbb{Z}_2$ these are of the Chebyshev type

$$\Pi_Q^{(L+4)} - 4c\Pi_Q^{(L+2)} + 4\Pi_Q^{(L)} = 0, \tag{11}$$

and so for $N = 2$ the $\Pi_Q^{(L)}$ form orthogonal sequences. However, for $\mathbb{Z}_3$ we have[11] (valid for all $L \geq 0$ and all Q):

$$\Pi_Q^{(L+9)} - 3(9c^2 - 5)\Pi_Q^{(L+6)} + 48\,\Pi_Q^{(L+3)} - 64\,\Pi_Q^{(L)} = 0. \tag{12}$$

These $\Pi_Q^{(L)}$ form 9 sequences, each labeled by (L_0, Q), where $Q = 0, 1, 2$ and $L = 3j + L_0$ where $L_0 = 0, 1, 2$, $j = 0, 1, 2, \dots$. The degrees of the polynomials appearing in this relation increase by two from the right to the left, but since the recursion is four-term, not three-term, these are not orthogonal sequences[21a]. However, like (11) also (12) are of the most simple type[b]: all coefficients are independent of L.

4. Expansion in terms of Jacobi polynomials

As the zeros of our $\Pi_Q^{(L)}$ are confined to and dense in the interval $(-1, +1)$ we call this the basic interval like for orthogonal polynomials. Trying to determine (numerically) a weight function by the ansatz $\int_{-1}^{1}(1+c)^{\alpha}(1-c)^{\beta}c^k\Pi_Q^{(L)}(c) = 0$ for $k < b_{Q,L}$ fitting α and β, we find (in the following we concentrate on the $\mathbb{Z}_3$-case) that there is an approximate solution very close to $\alpha = -\beta = \frac{1}{3}$, but α and β come out to be slightly L and k-dependent, in contrast to what is needed for orthogonality. However, for $L \to \infty$ and small fixed k, the solutions converge towards $\alpha = -\beta = \frac{1}{3}$. So the $\Pi_Q^{(L)}$ seem to be close to Jacobi polynomials $P_k^{(\frac{1}{3},-\frac{1}{3})}$, but can we formulate an exact relation valid for finite L? Is there an exact property of the $\Pi_Q^{(L)}$ which replaces orthogonality?

Numerical calculations[11] gave the surprising result that seemingly complicated $\Pi_Q^{(L)}$

[a] If we consider $Q = L \mod 3$, then we have only polynomials in c^2, and the degrees $b_{L,Q}$ are consecutive in powers of $z = c^2$ ("simple sets of polynomials") with integer coefficients.

[b] For higher N we get similar $N+1$-term relations, e.g. for $\mathbb{Z}_4$:
$\Pi_Q^{(L+16)} - 4(64c^3 - 56c)\Pi_0^{(4)}\Pi_Q^{(L+12)} - 128(14c^2 - 17)\Pi_Q^{(L+8)} - 2048c\Pi_Q^{(L+4)} + 4096\,\Pi_Q^{(L)} = 0.$

Table 1. Examples of $\mathbb{Z}_3$–polynomials $\Pi_Q^{(L)}(c)$ and their Jacobi-components.

	$\mathbb{Z}_3$, $Q = L+1 \mod 3$, $(\alpha,\beta) = (\frac{1}{3}, -\frac{1}{3})$	
L	$\Pi_Q^{(L)}(c)$	$2^{-[2L/3]}\,\Pi_Q^{(L)}(c)$
3	$9c+3$	$[0, \frac{9}{4}]$
4	$27c^2-18c-5$	$[3, -\frac{27}{4}, \frac{9}{2}]$
5	$81c^3+27c^2-57c-11$	$[0, -\frac{3}{2}, 0, \frac{81}{20}]$
6	$3^5c^3+81c^2-135c-21$	$[0, 0, 0, \frac{243}{40}]$
7	$3^6c^4-2\cdot 3^5c^3-540c^2+270c+43$	$[3, -\frac{27}{4}, \frac{171}{14}, -\frac{729}{40}, \frac{729}{70}]$
8	$3^7c^5+3^6c^4-2754c^3-702c^2+711c+85$	$[0, 0, 0, -\frac{27}{5}, 0, \frac{243}{28}]$
9	$3^8c^5+3^7c^4-7290c^3-1782c^2+1593c+171$	$[0, 0, 0, -\frac{81}{40}, 0, \frac{729}{56}]$
10	$3^9c^6-2\cdot 3^8c^5-25515c^4+14580c^3$ $+7965c^2-3186c-341$	$[3, -\frac{27}{4}, \frac{135}{14}, -\frac{243}{20}, \frac{9477}{385}, -\frac{3^7}{56}, \frac{3^8}{308}]$

can be written as a combination of just very few Jacobi polynomials, e.g. $\Pi_1^{(12)} = 3^{11}c^7+3^{10}c^6-5\cdot 3^{10}c^5-80919c^4+140697c^3+27459c^2-16839c-1365 = \frac{6^8}{728}\left(\frac{63}{22}\,P_7^{(\frac{1}{3},-\frac{1}{3})} - P_5^{(\frac{1}{3},-\frac{1}{3})}\right)$. For polynomials $\pi(c)$ of degree n we use[c]:

$$\pi(c) \;=\; [\,\pi_0,\, \pi_1,\, \ldots,\, \pi_n\,] \;\equiv\; \sum_{k=0}^{n} \pi_k\, P_k^{(\frac{1}{3},-\frac{1}{3})}(c) \qquad (13)$$

and define a scalar product with Baxter's variable $t = ((1-c)/(1+c))^{1/3}$ (see (6)) as the weight function (here we will not need to specify the normalization):

$$\langle \pi^{(1)} | \pi^{(2)} \rangle = \int_{-1}^{+1} dc \left(\frac{1-c}{1+c}\right)^{1/3} \pi^{(1)}(c)\pi^{(2)}(c) = -\int_{-\infty}^{0} dt \frac{6s}{(1-s)^2} \pi^{(1)}(c(s))\,\pi^{(2)}(c(s)).$$

The second part of this definition shows that it makes sense also if we prefer to use polynomials in the variable s, and that it preserves the original $\mathbb{Z}_3$–symmetry.

Since the $\Pi_Q^{(L)}$ satisfy the recursion relations (12) they can be written as determinants of band matrices with a bottom line specifying the initial conditions (which are the lowest L polynomials). Omitting the bottom line, we define $j \times j$–band matrices and their determinants $R_j = det\,|band([64, 48, 3p, 1, 0], j)|$, e.g.

$$R_6 \;=\; \begin{vmatrix} 3p & 1 & 0 & 0 & 0 & 0 \\ 48 & 3p & 1 & 0 & 0 & 0 \\ 64 & 48 & 3p & 1 & 0 & 0 \\ 0 & 64 & 48 & 3p & 1 & 0 \\ 0 & 0 & 64 & 48 & 3p & 1 \\ 0 & 0 & 0 & 64 & 48 & 3p \end{vmatrix} .$$

where $p = 9c^2-5$, so that the polynomials R_j depend only on c^2. Now we get the nine sequences of the $\Pi_Q^{(L)}$ as linear combinations of the R_j which satisfy appropriate

[c]For Jacobi series as a generalization of Taylor series, see Ch.7 of Carlson.[23] We use the standard normalization of Jacobi polynomials, see e.g. Rainville.[22]

initial conditions. Abbreviating $Q_j = R_j + 8R_{j-1}$ these are found to be:

$$\begin{aligned}
\Pi_0^{(3j)} &= \tfrac{1}{3}\,(Q_j - 8Q_{j-1} + 16Q_{j-2})\,; & \Pi_{\substack{1\\2}}^{(3j)} &= 9cR_{j-1} \pm 3Q_{j-1}; \\
\Pi_{\substack{0\\2}}^{(3j+1)} &= \pm 18cR_{j-1} + Q_j + 2Q_{j-1}; & \Pi_1^{(3j+1)} &= Q_j - 4Q_{j-1}; \qquad (14)\\
\Pi_{\substack{0\\1}}^{(3j+2)} &= 3c(R_j - 4R_{j-1}) \pm (Q_j - 4Q_{j-1}); & \Pi_2^{(3j+2)} &= 3Q_j.
\end{aligned}$$

For $j = 1, 2$ use $R_0 = 1$; $R_{-1} = R_{-2} = 0$. From (14) we see that only two of the 9 sequences are independent. There are relations like e.g. $\Pi_0^{(3j+1)} - \Pi_1^{(3j+1)} = 2\Pi_1^{(3j)}$.

To get the Jacobi-expansion of the $\Pi_Q^{(L)}$, we only need to expand R_j and cR_j. Using Jacobi-components defined in (13), from explicit calculation (for $j \le 36$), we find that for $k < j$ (*only there*) the j-dependence of the $(R_j)_k$ obeys the simple rule:

$$(R_j)_k \;= (-3)^k\, k!\,(2k+1)\left\{(-8)^j\,\sigma_k \;+\; 4^j\,\tau_k\right\};$$

$$\tau_k = \frac{1}{3\prod_{n=0}^{k}(3n+1)}; \qquad \sigma_{k=2m} = \frac{2\,(-3)^m}{3\prod_{n=m}^{3m}(2n+1)}; \qquad \sigma_{k=2m-1} = \frac{2\,(-3)^{m-2}}{\prod_{n=m}^{3m-1}(2n)}.$$

It follows that the σ_k do not contribute to the $k < j$-components of Q_j, and, using the recursion relations for the $P_k^{(\frac{1}{3},-\frac{1}{3})}(c)$, we conclude that for $k \le j$ we have $\frac{1}{3}(Q_j)_k = -(cR_j)_k = \frac{1}{4}(c^2R_{j+1})_k = 4^j(-3)^k\,k!\,(2k+1)\tau_k$. It further follows that

$$(\Pi_0^{(3j+3)})_k = (\Pi_1^{(3j)})_k = (\Pi_0^{(3j+1)})_k = (\Pi_{\substack{0\\1}}^{(3j+1)})_k = (\Pi_{\substack{0\\1}}^{(3j+2)})_k = 0 \qquad \text{for} \quad k < j.$$

All $k < j$ components of $\Pi_2^{(3j)}$, $\Pi_2^{(3j+1)}$ and $\Pi_2^{(3j+2)}$ are proportional to $(Q_j)_k$. We get zero overlap between a polynomial $\Pi_Q^{(L)}$ of degree $b_{L,Q}$ with all polynomials $\Pi_{Q'}^{(L')}$ which have at least $b_{L,Q}$ vanishing low-k components (these can only be polynomials which have $b_{L',Q'} > 2\,b_{L,Q}$). One of many such relations is e.g.

$$\langle\, \Pi_Q^{(3j)} \mid \Pi_{Q'}^{(3j')} \,\rangle = 0 \quad \text{for} \quad Q = Q' = 1 \qquad \text{and} \quad 2j \le j' - 1.$$

This property may be called "partial orthogonality".

Further rules, this time valid for all k, regard the vanishing of many components for particular linear combinations: Defining

$$\begin{aligned}
\mathcal{Q}_+^{(j)} &\equiv \Pi_1^{(3j)} = 9\,c\,R_{j-1} + 3\,Q_{j-1} = [\,\alpha_0^{(+)},\ \alpha_1^{(+)},\ \ldots,\ \alpha_{2j-1}^{(+)}\,];\\
\mathcal{Q}_-^{(j)} &\equiv \quad Q_j + 9\,c\,R_{j-1} - Q_{j-1} \quad = [\,\alpha_0^{(-)},\ \alpha_1^{(-)},\ \ldots,\ \alpha_{2j}^{(-)}\,],
\end{aligned}$$

we have checked up to $j = 30$ that $\alpha_k^{(\pm)} = 0$ for $k < j$ and that all even (odd) k Jacobi components of $\mathcal{Q}_+^{(j)}$ ($\mathcal{Q}_-^{(j)}$) vanish. So we conjecture for all j, j'

$$\langle\, \mathcal{Q}_+^{(j)} \mid \mathcal{Q}_-^{(j')} \,\rangle = 0.$$

One can check some special cases of these results in Table 1. It has been found numerically[11] that a similar partial orthogonality appears also for the $\mathbb{Z}_N$-Baxter polynomials with $N = 4$, 5, 6. For even values of N further relations emerge, but these will not be discussed here.

5. Conclusion

The polynomials which play a central role for the calculation of the energy eigenvalues of the superintegrable $\mathbb{Z}_3-$chiral Potts model are found to be related to Jacobi polynomials in a very peculiar way. Many integrals giving the Jacobi-coefficients of Baxter's polynomials are found to vanish. These observations should have a deeper group-theoretical background, but the underlying symmetry is not yet clear to us. By reducing the formulation of the problem to some basic facts, the present analysis tries to prepare the ground for clarifying the symmetry involved.

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THERMODYNAMICS OF TWO COMPONENT BOSONS IN ONE DIMENSION

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On the basis of Bethe ansatz solution of two-component bosons with SU(2) symmetry and δ-function interaction in one dimension, we study the thermodynamics of the system at finite temperature by using the strategy of thermodynamic Bethe ansatz (TBA). It is shown that the ground state is an isospin "ferromagnetic" state by the method of TBA, and at high temperature the magnetic property is dominated by Curie's law. We obtain the exact result of specific heat and entropy in strong coupling limit which scales like T at low temperature. While in weak coupling limit, it is found there is still no Bose-Einstein Condensation (BEC) in such 1D system.

1. Introduction

A two-component Bose gas has been produced in magnetically trapped ^{87}Rb by rotating the two hyperfine states into each other with the help of slightly detuned Rabi oscillation field.[1] It was noticed[2] that the properties of such Bose system can be different from the traditional scalar Bose system once it acquires internal degree of freedom. Bethe ansatz solution of SU(2) two-component bosons in one dimension was obtained.[3,4] It was pointed out that the ground state of such a system is an isospin "ferromagnetic" state[4] which differs from that of spin-1/2 fermions in one dimension, such as SU(2) Hubbard model,[5] etc.

An interacting SU(2) boson field trapped in a one dimensional ring of length L

can be modeled by the following Hamiltonian

$$H = \int dx \left[\sum_a \partial_x \psi_a^* \partial_x \psi_a + \frac{c}{2} \sum_{a,b} \psi_a^* \psi_a \psi_b^* \psi_b \right] \tag{1}$$

where $a, b = 1, 2$ denotes the z-component of isospin. The Bethe ansatz equations (BAE) of eq. (1) are obtained as follows

$$\begin{aligned} e^{ik_j L} &= -\prod_{l=1}^{N} \frac{k_j - k_l + ic}{k_j - k_l - ic} \prod_{\nu=1}^{M} \frac{k_j - \lambda_\nu - ic/2}{k_j - \lambda_\nu + ic/2} \\ 1 &= -\prod_{l=1}^{N} \frac{\lambda_\gamma - k_l - ic/2}{\lambda_\gamma - k_l + ic/2} \prod_{\nu=1}^{M} \frac{\lambda_\gamma - \lambda_\nu + ic}{\lambda_\gamma - \lambda_\nu - ic} \end{aligned} \tag{2}$$

where M denotes the total number of down isospins. Eq. (2) differs from the BAE of scalar bosons with periodic condition.[6] The second equation of eqs. (2) of isospin rapidity λ arises from the application of quantum inverse method,[7] which can be inferred from spin-1/2 fermions[8] too. However, the symmetry of bosonic wave function gives the first term on the right side of first equation of eqs. (2), which does not appear in the BAE for fermions.

2. Thermodynamics at Finite Temperature

The strategy we use here is the thermodynamic Bethe ansatz (TBA) which was pioneered by C. N. Yang and C. P. Yang for the case of the delta-function Bose gas.[10] It is used to derive a set of nonlinear integral equations called TBA equations, which describe the thermodynamics of the model at finite temperature. Moreover, the λs can be complex roots which should form a "bound state" with other λs[11] when $T \neq 0$, which arises from the consistency of both sides of the BAE. For ideal λ strings of length m the rapidities are $\Lambda_a^{nj} = \lambda_a^n + (n+1-2j)iu + O(\exp(-\delta N))$. Here $u = c/2$, a enumerates the strings of the same length m, and $j = 1, \ldots, m$ counts the λs involved in the ath λ string of the length m, λ_a^n is the real part of the string.

Taking logarithm of the BAE (2) by using string hypothesis we arrive at the following discrete Bethe ansatz equations

$$\begin{aligned} 2\pi I_j &= k_j L + 2 \sum_l \Theta_2(k_j - k_l) - \sum_{an} \Theta_n(k_j - \lambda_a^n) \\ 2\pi J_a^n &= 2 \sum_l \Theta_n(\lambda_a^n - k_l) - 2 \sum_{bl, t \neq 0} A_{nlt} \Theta_t(\lambda_a^n - \lambda_b^l) \end{aligned} \tag{3}$$

where $\Theta_n(x) = \tan^{-1}(x/nu)$ and

$$A_{nlt} = \begin{cases} 1, \text{ for } t = n + l, |n - l| \\ 2, \text{ for } t = n + l - 2, \cdots, |n - l| + 2 \\ 0, \text{ otherwise.} \end{cases}$$

I_j and J_a^n play the role of quantum numbers for charge rapidity and isospin rapidity respectively. In order to guarantee linear independence of wave function, all quantum number within a given set of $\{I\}$ as well as that in $\{J\}$ should be different. An arbitrary quantum number may be either in the set or not in the set. The former is called a root, the later is called a hole. In thermodynamic limit, the distribution of charge rapidities becomes dense and it is useful to introduce the density function for charge roots and holes respectively. We denote with $\rho(k)$ and $\rho^h(k)$ the density function of charge roots and holes, in a similar way, with $\sigma_n(\lambda)$ and σ_n^h the density function of n-strings roots and holes on real axis. They are defined by

$$\begin{aligned} \rho(k) + \rho^h(k) &= (1/L)dI(k)/dk \\ \sigma_n(\lambda) + \sigma_n^h(\lambda) &= (1/L)dJ^n(\lambda)/d\lambda. \end{aligned} \tag{4}$$

Then from eqs. (3) we obtain a set of coupled integral equations.

$$\begin{aligned} \rho + \rho^h &= \frac{1}{2\pi} + K_2(k) * \rho(k) - \sum_n K_n(k) * \sigma_n(k) \\ \sigma_n + \sigma_n^h &= K_n(\lambda) * \rho(\lambda) - \sum_{l,t\neq 0} A_{nlt} K_t(\lambda) * \sigma_l(\lambda) \end{aligned} \tag{5}$$

where $K_n(x) = nu/\pi(n^2u^2 + x^2)$, and $*$ denotes the integral convolution.

In terms of the density functions of charge and isospin roots, the kinetic energy per length has the form $E_k/L = \int k^2\rho(k)dk$, the total number of down isospins is $M/L = \sum_n n \int \sigma_n(\lambda)d\lambda$ and the particle density of the model is $D = N/L = \int \rho(k)dk$. If we consider the energy arising from the external field Ω which is the Rabi field in two-component BEC experiments, the internal energy of the model is

$$E/L = \int (k^2 - \Omega)\rho(k)dk + \sum_n 2n\Omega \int \sigma_n d\lambda. \tag{6}$$

And with the help of the approach first introduced by Yang and Yang,[10] the entropy of the present model at finite temperature is

$$\begin{aligned} \mathcal{S}/L = &\int [(\rho + \rho^h)\ln(\rho + \rho^h) - \rho\ln\rho - \rho^h \ln\rho^h]dk \\ &+ \sum_n \int [(\sigma_n + \sigma_n^h)\ln(\sigma_n + \sigma_n^h) - \sigma_n \ln\sigma_n - \sigma_n^h \ln\sigma_n^h]d\lambda. \end{aligned} \tag{7}$$

The Gibbs free energy of the model then is defined by $F = E - T\mathcal{S} - \mu N$, where μ is the chemical potential. In order to obtain the thermal equilibrium, we minimize the free energy with respect to all the density functions subjects to the constraint (5). In addition, the total number of particles, the magnetization are kept to constant. For this purpose, the chemical potential μ and external field Ω play the role of Lagrange multipliers.

It is useful to define the energy potential for charge sector and isospin sector:

$$\begin{aligned} \kappa(k) &= e^{\epsilon(k)/T} = \rho^h(k)/\rho(k) \\ \eta_n(\lambda) &= \sigma_n^h(\lambda)/\sigma_n(\lambda). \end{aligned} \tag{8}$$

Applying the minimum condition $\delta F = 0$ gives rise to a revised version of Gaudin-Takahashi equations

$$\begin{aligned} T\ln\kappa = \ & \epsilon(k) = k^2 - \mu - \Omega - TK_2(k) * \ln[1+\kappa^{-1}] \\ & -T\sum_n K_n(k) * \ln[1+\eta^{-1}] \\ \ln\eta_1 = \ & \frac{1}{4u}\mathrm{sech}(\pi\lambda/2u) * \ln[(1+\kappa^{-1})(1+\eta_2)] \\ \ln\eta_n = \ & \frac{1}{4u}\mathrm{sech}(\pi\lambda/2u) * \ln[(1+\eta_{n-1})(1+\eta_{n+1})]. \end{aligned} \tag{9}$$

And these equations are completed by the asymptotic conditions

$$\lim_{n\to\infty}[\ln\eta_n/n] = 2x \tag{10}$$

where $x = \Omega/T$. Eqs. (9) can be solved by iteration. Note that eqs. (5) together with eqs. (9) completely determine the densities of charge roots and isospin roots in the state of thermal equilibrium. The Helmholtz free energy $F = E - TS$ is given by

$$F = \mu N - \frac{TL}{\pi}\int \ln[1+e^{-\epsilon/T}]dk. \tag{11}$$

The above approach called TBA is universal for discussing the thermodynamics of one dimensional integrable model. Once eqs. (9) are solved, all thermodynamic quantities can be obtained from eq. (11) in principle.

3. Magnetic Property: Zero and High Temperature Limit:

The state at zero temperature is the ground state. The Fermi surface is determined by $\epsilon(k_F) = 0$. Since there is no hole under Fermi surface, we can take the energy potential $\kappa = \rho^h/\rho$ as zero. As a result, from eqs. (9), it is easy to see $\eta_n \to \infty$, and $M = 0$, the "ferromagnetic" ground state. The first equation of eqs. (9) becomes

$$\epsilon_0(k) = k^2 - \mu - \Omega + K_2(k) * \epsilon_0(k) \tag{12}$$

which gives the solution of dressed energy, and the ground-state energy may be given in terms of ϵ_0

$$E_0/L = \frac{1}{2\pi}\int_{-k_F}^{k_F}\epsilon_0(k)dk. \tag{13}$$

Consequently, the ground state of 1D SU(2) bosons is an isospin "ferromagnetic" state, which coincides with the analysis of Li et al.[4] Then the property of the model at $T = 0$ is the same as that of scalar bosons in one dimension which has been discussed extensively by Lieb and Liniger.[6] In the isospin space, however, the SU(2) symmetry of whole system around the ground state is broken.

In the high temperature limit $T \to \infty$ (free isospins), however we can assume that all functions $\eta_n(\lambda)$ are independent of λ. Then eqs. (9) can be written as follows,

$$\begin{aligned} \eta_1^2 &= (1+\eta_2) \\ \eta_n^2 &= (1+\eta_{n-1})(1+\eta_{n+1}) \end{aligned} \tag{14}$$

where we have neglected the term $(1+\kappa^{-1})$ in the second equation of eqs. (9). The solution of η_n are then constants fixed by the field boundary condition (10) to be

$$\eta_n = \left[\frac{\sinh(n+1)x}{\sinh x}\right]^2 - 1. \tag{15}$$

After perform the Fourier transformation on eqs. (5), we get the solution of the densities of λ n-strings,

$$\begin{aligned} \sigma_1 + \sigma_1^h &= \frac{1}{4u}\mathrm{sech}[\pi\lambda/2u] * [\rho + \sigma_2^h] \\ \sigma_n + \sigma_n^h &= \frac{1}{4u}\mathrm{sech}[\pi\lambda/2u] * [\sigma_{n+1}^h + \sigma_{n-1}^h]. \end{aligned} \tag{16}$$

If we assume that σ_n and σ_n^h are independent of λ or let $c = 0$, the total number of down isospins has the form,

$$\sum_n n\sigma_n = \frac{\rho}{2} - \frac{n_m+1}{2}\sigma_{n_m} e^{n_m\Omega/T} \tag{17}$$

where n_m is maximal length of λ strings. In the absence of Rabi field, we have $M/N = 1/2$, the system at high temperature is a quasi "paramagnetic" state. If the external field Ω is small, expanding eq. (17) for small field x and integrating the equation over λ space, we get the magnetization of the model. Let M_m be the total number of isospin rapidities in all n_m-strings,

$$\frac{S_z}{L} = \frac{M_m}{2L}\left(1 + \frac{n_m\Omega}{T} + \frac{n_m^2\Omega^2}{2T^2} + \cdots\right) \tag{18}$$

where the first term in the parentheses arises from self-magnetization, while the others are contributed by Rabi field. Eq. (18) indicates that the magnetic property of the model in high temperature regime dominated by Curie's law $\chi \propto 1/T$.

4. Strong and Weak Coupling Limit:

When $\eta \to \infty$, $K_n(k) = 0$, from eqs. (9) we have

$$\epsilon = k^2 - \Omega - \mu. \tag{19}$$

The free energy of the system (11) at low temperature now can be solved by integration by part,

$$F/L = \mu D - \frac{2}{\pi}\left[\frac{1}{3}\mu^{3/2} + \frac{T^2\pi^2}{24\mu^{1/2}}\right] \tag{20}$$

where the external field is set to zero.

We can not deduce the specific heat directly from the free energy obtained above because the chemical potential is a function of temperature. From eqs. (5), the density of charge rapidity has the form

$$\rho = \frac{1}{2\pi}\frac{1}{1+e^{(k^2-\mu)/T}}. \tag{21}$$

Clearly, at zero temperature, the Fermi surface is just the square root of the chemical potential, so we have $\mu_0 = \pi^2 D^2$. At low temperature, however, it is determined by $D = N/L = \int \rho(k)dk$. After integration, we have a temperature dependent chemical potential

$$\mu = \mu_0\Big[1 - \frac{\pi^2 T^2}{24\mu_0^2}\Big]^{-2}. \tag{22}$$

Then the free energy becomes

$$F/L = \mu_0 D\Big[1 + \frac{\pi^2 T^2}{12\mu_0^2}\Big] - \frac{2}{3\pi}\mu_0^{3/2}\Big[1 + \frac{\pi^2 T^2}{4\mu_0^2}\Big]. \tag{23}$$

Since by thermodynamics $\mathcal{S} = -\partial F/\partial T$ and $C_v = T\partial\mathcal{S}/\partial T$, we find the specific heat at low temperature is Fermi-liquid like

$$\mathcal{S} = C_v = \frac{T}{6D}. \tag{24}$$

It is the same as the result of one-component case, since for the strong coupling limit the isospin and the charge are decoupled completely, the contribution of isospin to the free energy vanishes.

In order to discuss the possibility of the existence of BEC, we consider the problem in weak coupling limit $u \to 0$. And isospin-isospin reaches its maximal correlation. At low temperature, however, we do not take string hypothesis for simplicity. Because $\lim_{c\to 0} K_n(x) = \delta(x)$, together with eqs. (5) and eqs. (9), we obtain

$$\rho(k) = \frac{1}{2\pi}\frac{(3e^{\varepsilon_0}-1)(e^{-\varepsilon_0}+1)}{(3e^{2\varepsilon_0}+1)(1-e^{-\varepsilon_0})} \tag{25}$$

where $\varepsilon_0 = (k^2-\mu)/T$. The positive definition of $\rho(k)$ requires that the chemical potential is negative. As we known the density of scalar boson is $2\pi\rho = 1/(1-e^{-\varepsilon_0})$ which prevents the BEC in 1D and 2D system because of the infrared divergence. However, the density function (25) still does not resolve this problem. Consequently, BEC does not happen in this model yet.

5. Conclusion and Acknowledgment

To summarize, we discussed the general thermodynamics of one dimensional SU(2) bosons with δ-function interaction by using the strategy of TBA. It was shown that the ground state is an isospin "ferromagnetic" state which differs from the ground state of 1D fermions, while at high temperature, it is "paramagnetic" state and

the magnetic property is dominated by Curie's law. In strong coupling limit, we obtain the exact expression of the dependence of chemical potential, entropy and specific heat on temperature which are Fermi-liquid like, while in weak coupling limit, we found the infrared divergence of charge roots density function prevents the existence of BEC.

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R-MATRICES AND THE TENSOR PRODUCT GRAPH METHOD

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A systematic method for constructing trigonometric R-matrices corresponding to the (multiplicity-free) tensor product of any two affinizable representations of a quantum algebra or superalgebra has been developed by the Brisbane group and its collaborators. This method has been referred to as the Tensor Product Graph Method. Here we describe applications of this method to untwisted and twisted quantum affine superalgebras.

1. Introduction

The (graded) Yang-Baxter equation (YBE) plays a central role in the theory of (supersymmetric) quantum integrable systems. Solutions to the YBE are usually called R-matrices. The knowledge of R-matrices has many physical applications. In one-dimensional lattice models, R-matrices yield the Hamiltonians of quantum spin chains.[1] In statistical mechanics, R-matrices define the Boltzmann weights of exactly soluble models[2] and in integrable quantum field theory they give rise to exact factorizable scattering S-matrices.[3] So the construction of R-matrices is fundamental in the study of integrable systems.

Mathematical structures underlying the YBE and therefore R-matrices and integrable models are quantum affine (super)algebras. A systematic method for the construction of trigonometric R-matrices arising from untwisted and twisted quantum affine (super)algebras has been developed in Refs. 4–9 (see also Ref. 10 for rational cases). This method is called the Tensor Product Graph (TPG) method. The method enables one to construct spectral dependent R-matrices corresponding to the (multiplicity-free) tensor product of *any* two affinizable representations of a quantum algebra or superalgebra.

In this contribution, we describe the TPG method in the context of untwisted and twisted quantum affine superalgebras. Quantum superalgebras are interesting since the tensor product decomposition often has indecomposables and integrable models associated with them may in some instances be interpreted as describing strongly correlated fermion systems.[11,12]

2. Quantum Affine Superalgebras and Jimbo Equation

Let us first of all recall some facts about the affine superalgebra $\mathcal{G}^{(k)}$, $k = 1, 2$. Let $\mathcal{G}_0$ be the fixed point subalgebra under the diagram automorphism of $\mathcal{G}$ of order k. In the case of $k = 1$, we have $\mathcal{G}_0 \equiv \mathcal{G}$. For $k = 2$ we may decompose $\mathcal{G}$ as $\mathcal{G}_0 \oplus \mathcal{G}_1$, where $[\mathcal{G}_0, \mathcal{G}_1] \subset \mathcal{G}_1$. Let ψ be the highest root of $\mathcal{G}_0 \equiv \mathcal{G}$ for $k = 1$ and θ be the highest weight of the $\mathcal{G}_0$-representation $\mathcal{G}_1$ for $k = 2$.

Quantum affine superalgebras $U_q[\mathcal{G}^{(k)}]$ are q-deformations of the universal enveloping algebras $U[\mathcal{G}^{(k)}]$ of $\mathcal{G}^{(k)}$. We shall not give the defining relations for $U_q[\mathcal{G}^{(k)}]$, but mention that the action of the coproduct on its generators $\{h_i,\ e_i,\ f_i,\ 0 \leq i \leq r\}$ is given by

$$\begin{aligned} &\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \\ &\Delta(e_i) = e_i \otimes q^{\frac{h_i}{2}} + q^{-\frac{h_i}{2}} \otimes e_i, \qquad \Delta(f_i) = f_i \otimes q^{\frac{h_i}{2}} + q^{-\frac{h_i}{2}} \otimes f_i. \end{aligned} \tag{1}$$

Define an automorphism D_z of $U_q[\mathcal{G}^{(k)}]$ by

$$D_z(e_i) = z^{k\delta_{i0}} e_i, \qquad D_z(f_i) = z^{-k\delta_{i0}} f_i, \qquad D_z(h_i) = h_i. \tag{2}$$

Given any two minimal irreducible representations π_λ and π_μ of $U_q[\mathcal{G}_0]$ and their affinizations to irreducible representations of $U_q[\mathcal{G}^{(k)}]$, we obtain a one-parameter family of representations $\Delta^z_{\lambda\mu}$ of $U_q[\mathcal{G}^{(k)}]$ on $V(\lambda) \otimes V(\mu)$ defined by

$$\Delta^z_{\lambda\mu}(a) = \pi_\lambda \otimes \pi_\mu \left((D_z \otimes 1)\Delta(a) \right), \quad \forall a \in U_q[\mathcal{G}^{(k)}], \tag{3}$$

where z is the spectral parameter. Let $R^{\lambda\mu}(z)$ be the spectral dependent R-matrices associated with π_λ and π_μ, which satisfies the YBE. Moreover it obeys the intertwining properties:

$$R^{\lambda\mu}(z)\, \Delta^z_{\lambda\mu}(a) = (\Delta^T)^z_{\lambda\mu}(a)\, R^{\lambda\mu}(z) \tag{4}$$

which, according to Jimbo,[13] uniquely determine $R^{\lambda\mu}(z)$ up to a scalar function of z. We normalize $R^{\lambda\mu}(z)$ such that $\check{R}^{\lambda\mu}(z)\check{R}^{\mu\lambda}(z^{-1}) = I$, where $\check{R}^{\lambda\mu}(z) = P\, R^{\lambda\mu}(z)$ with $P : V(\lambda) \otimes V(\mu) \to V(\mu) \otimes V(\lambda)$ the usual graded permutation operator.

In order for the equation (4) to hold for all $a \in U_q[\mathcal{G}^{(k)}]$ it is sufficient that it holds for all $a \in U_q(\hat{L}_0)$ and in addition for the extra generator e_0. The relation for e_0 reads explicitly

$$\begin{aligned} &\check{R}^{\lambda\mu}(z) \left(z\, \pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) + \pi_\lambda(q^{-h_0/2}) \otimes \pi_\mu(e_0) \right) \\ &\quad = \left(\pi_\mu(e_0) \otimes \pi_\lambda(q^{h_0/2}) + z\, \pi_\mu(q^{-h_0/2}) \otimes \pi_\lambda(e_0) \right) \check{R}^{\lambda\mu}(z). \end{aligned} \tag{5}$$

Eq.(5) is the Jimbo equation for $U_q[\mathcal{G}^{(k)}]$.

3. Solutions to Jimbo Equation and Tensor Product Graph Method

Let $V(\lambda)$ and $V(\mu)$ denote any two minimal irreducible representations of $U_q[\mathcal{G}^{(k)}]$. Assume the tensor product module $V(\lambda) \otimes V(\mu)$ is completely reducible into irre-

ducible $U_q[\mathcal{G}_0]$-modules as

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu} V(\nu) \tag{6}$$

and there are no multiplicities in this decomposition. We denote by $P_\nu^{\lambda\mu}$ the projection operator of $V(\lambda) \otimes V(\mu)$ onto $V(\nu)$ and set $\mathbf{P}_\nu^{\lambda\mu} = \check{R}^{\lambda\mu}(1)\, P_\nu^{\lambda\mu} = P_\nu^{\mu\lambda}\, \check{R}^{\lambda\mu}(1)$. We may thus write

$$\check{R}^{\lambda\mu}(z) = \sum_{\nu} \rho_\nu(z)\, \mathbf{P}_\nu^{\lambda\mu}, \quad \rho_\nu(1) = 1. \tag{7}$$

Following our previous approach,[5] the coefficients $\rho_\nu(z)$ may be determined according to the recursion relation

$$\rho_\nu(z) = \frac{q^{C(\nu)/2} + \epsilon_\nu \epsilon_{\nu'} z\, q^{C(\nu')/2}}{z\, q^{C(\nu)/2} + \epsilon_\nu \epsilon_{\nu'}\, q^{C(\nu')/2}} \rho_{\nu'}(z), \tag{8}$$

which holds for any $\nu \neq \nu'$ for which

$$P_\nu^{\lambda\mu} \left(\pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) \right) P_{\nu'}^{\lambda\mu} \neq 0. \tag{9}$$

Here $C(\nu)$ is the eigenvalue of the universal Casimir element of $\mathcal{G}_0$ on $V(\nu)$ and ϵ_ν denotes the parity of $V(\nu) \subseteq V(\lambda) \otimes V(\mu)$.

We note that $e_0 \otimes q^{h_0/2}$ transforms under the adjoint action of $U_q[\mathcal{G}_0]$ as the lowest weight of $\mathcal{G}_0$-module $V(\psi)$ [resp. $V(\theta)$] for $k = 1$ (resp. $k = 2$) (i.e. as the lowest component of a tensor operator). Throughout we adopt the notation

$$< a >_{\pm} = \frac{1 \pm z\, q^a}{z \pm q^a}, \tag{10}$$

so that the relation (8) may be expressed as

$$\rho_\nu(z) = \left\langle \frac{C(\nu') - C(\nu)}{2} \right\rangle_{\epsilon_\nu \epsilon_{\nu'}} \rho_{\nu'}(z). \tag{11}$$

To graphically encode the recursion relations between different ρ_ν we introduce the **Extended TPG** for $U_q[\mathcal{G}^{(1)}]$ and **Extended Twisted TPG** for $U_q[\mathcal{G}^{(2)}]$.

Definition 1: The **Extended TPG** associated to the tensor product $V(\lambda) \otimes V(\mu)$ is a graph whose vertices are the irreducible modules $V(\nu)$ appearing in the decomposition (6) of $V(\lambda) \otimes V(\mu)$. There is an edge between two vertices $V(\nu)$ and $V(\nu')$ iff

$$V(\nu') \subset V_{adj} \otimes V(\nu) \quad \text{and} \quad \epsilon(\nu)\epsilon(\nu') = -1. \tag{12}$$

The condition (12) is a necessary condition for (9) corresponding to $U_q[\mathcal{G}^{(1)}]$ to hold.

Definition 2: The **Extended Twisted TPG** which has the same set of nodes as the twisted TPG but has an edge between two vertices $\nu \neq \nu'$ whenever

$$V(\nu') \subseteq V(\theta) \otimes V(\nu) \tag{13}$$

and

$$\epsilon_\nu \epsilon_{\nu'} = \begin{cases} +1 & \text{if } V(\nu) \text{ and } V(\nu') \text{ are in the same irreducible representation of } \mathcal{G} \\ -1 & \text{if } V(\nu) \text{ and } V(\nu') \text{ are in different irreducible representations of } \mathcal{G}. \end{cases} \tag{14}$$

The conditions (13) and (14) are necessary conditions for (9) corresponding to $U_q[\mathcal{G}^{(2)}]$ to hold.

We will impose a relation (8) for every edge in the extended (twisted) TPG but we will be imposing too many relations in general. These relations may be inconsistent and we are therefore not guaranteed a solution. If however a solution to the recursion relations exists, then it must give the unique correct solution to the Jimbo's equation.

4. Examples of R-matrices for $U_q[gl(m|n)^{(1)}]$

Throughout we introduce $\{\epsilon_i\}_{i=1}^m$ and $\{\delta_j\}_{j=1}^n$ which satisfy $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\delta_i, \delta_j) = -\delta_{ij}$ and $(\epsilon_i, \delta_j) = 0$. As is well known, every irreducible representation of $U_q[gl(m|n)]$ provides also an irreducible representation for $U_q[gl(m|n)^{(1)}]$. Here, as examples, we will construct the R-matrices corresponding to the following tensor product: rank a antisymmetric tensor with rank b antisymmetric tensor of the same type. Without loss of generality, we assume $m \geq a \geq b$ and the antisymmetric tensors to be contravariant. The tensor product decomposition is

$$V(\lambda_a) \otimes V(\lambda_b) = \bigoplus_c V(\Lambda_c) \tag{15}$$

where, when $a + b \leq m$,

$$\lambda_b = \sum_{i=1}^{b} \epsilon_i\,, \qquad \Lambda_c = \sum_{i=1}^{a+c} \epsilon_i + \sum_{i=1}^{b-c} \epsilon_i, \qquad c = 0, 1, \cdots, b \tag{16}$$

and when $a + b > m$,

$$\begin{aligned} \Lambda_c &= \sum_{i=1}^{a+c} \epsilon_i + \sum_{i=1}^{b-c} \epsilon_i, \qquad c = 0, 1, \cdots, m-a \\ \Lambda_c &= \sum_{i=1}^{m} \epsilon_i + \sum_{i=1}^{b-c} \epsilon_i + (a+c-m)\delta_1, \qquad c = m-a+1, \cdots, b \end{aligned} \tag{17}$$

The corresponding TPG is

$V(\lambda_a) \otimes V(\lambda_b)$ =

(18)

which is consistent; such is always the case when a graph is a tree (i.e. contains no closed loops). From the graph we obtain

$$\check{R}^{\lambda_a,\lambda_b}(x) = \sum_{c=0}^{b} \prod_{i=1}^{c} \langle 2i + a - b \rangle_- \, \mathbf{P}_{\Lambda_c}^{\lambda_a,\lambda_b} \tag{19}$$

The $a = b = 1$ case had been worked out before, which is known to give rise to the Perk-Schultz model R-matrices.[14,15]

5. Examples of R-matrices for $U_q[gl(n|n)^{(2)}]$

To begin with, we introduce the concept of minimal representations. By minimal irreducible representations of $\mathcal{G}$, we mean those irreducible representations which are also irreducible under the fixed subalgebra $\mathcal{G}_0$. We can determine R-matrices for any tensor product $V(\lambda_a) \otimes V(\lambda_b)$ of two minimal representations $V(\lambda_a)$, $V(\lambda_b)$ of $U_q[gl(m|n)^{(2)}]$, where $V(\lambda_a)$ is also irreducible model under $U_q[osp(m|n)]$ with the corresponding $U_q[osp(m|n)]$ highest weight $\lambda_a = (\dot{0}|a, \dot{0})$. Recall that for our case $\mathcal{G}_0 \equiv osp(m|n)$ and $\theta = \delta_1 + \delta_2$. Below we shall illustrate the method for the interesting case of $a = b$, $m = n > 2$, where an indecomposable appears in the tensor product decomposition.

The decomposition of the tensor product of two minimal irreducible representations of $U_q[osp(m|n)]$:[9]

$$V(\lambda_a) \otimes V(\lambda_b) = \bigoplus_{c=0}^{a} \bigoplus_{k=0}^{c} V(k, a+b-2c); \tag{20}$$

here and throughout $V(a, b)$ denotes an irreducible $U_q[osp(m|n)]$ module with highest weight $\lambda_{a,b} = (\dot{0}|a+b, a, \dot{0})$. Note that one can only get an indecomposable in (20) when $m = n > 2$ and $a + b - 2c = 0$. Since $a \leq b$, $c \leq a$, this can only occur when $a = b$ and $c = a$. In that case the $U_q[osp(m|n)]$-modules $V(k, 0)$, $k = 0, 1$, will form an indecomposable. From now on we denote by V this indecomposable module, and write the $U_q[osp(n|n)]$ module decomposition (20) as

$$V(\lambda_a) \otimes V(\lambda_a) = \bigoplus_{\nu} V(\nu) \bigoplus V, \tag{21}$$

where the sum on ν is over the irreducible highest weights. Note that V contains a unique submodule $\bar{V}(\delta_1 + \delta_2)$ which is maximal, indecomposable and cyclically generated by a maximal vector of weight $\delta_1 + \delta_2$ such that $V/\bar{V}(\delta_1 + \delta_2) \cong V(\dot{0}|\dot{0})$ (the trivial $U_q[osp(n|n)]$-module). Moreover V contains a unique irreducible submodule $V(\dot{0}|\dot{0}) \subset \bar{V}(\delta_1 + \delta_2)$. The usual form of Schur's lemma applies to $\bar{V}(\delta_1 + \delta_2)$ and so the space of $U_q[osp(n|n)]$ invariants in $\mathrm{End}(V)$ has dimension 2. It is spanned by the identity operator I together with an invariant N (unique up to scalar multiples) satisfying

$$N\,V = V(\dot{0}|\dot{0}) \subset \bar{V}(\delta_1 + \delta_2), \qquad N\,\bar{V}(\delta_1 + \delta_2) = (0). \tag{22}$$

It follows that N is nilpotent, i.e. $N^2 = 0$.

We can show[9] that the minimal irreducible $U_q[osp(n|n)]$ modules, $V(\lambda_a)$, with highest weight λ_a, are affinizable to carry irreducible representations of $U_q[gl(n|n)^{(2)}]$. We now determine the extended twisted TPG for the decomposition given by (21).

We note that V can only be connected to two nodes corresponding to highest weights

$$\nu = \begin{cases} 2\delta_1 \ \text{(opposite parity)}, & (c,k) = (a-1,0) \\ 2(\delta_1+\delta_2) \ \text{(same parity)}, & (c,k) = (a,2). \end{cases} \tag{23}$$

We thus arrive at the extended twisted TPG for (21), given by Figure 1.

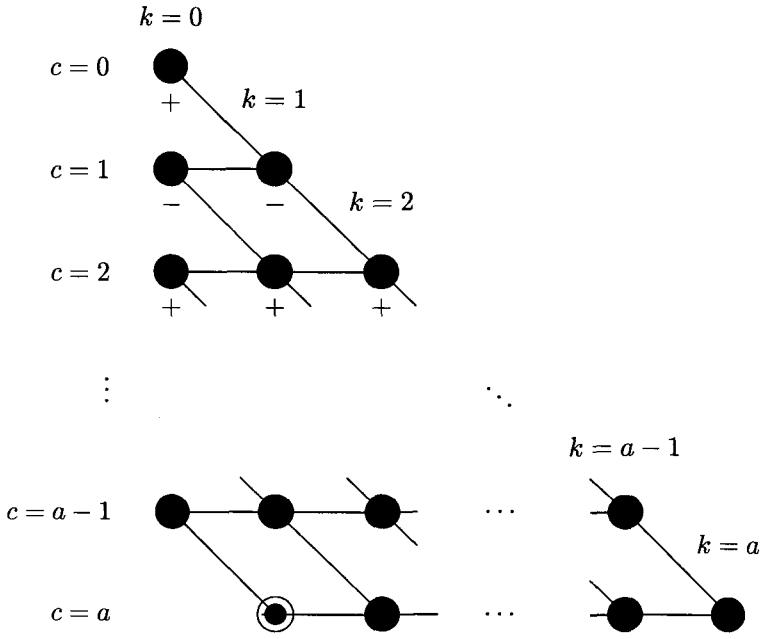

Fig. 1. The extended twisted TPG for $U_q[gl(n|n)^{(2)}]$ $(n > 2)$ for the tensor product $V(\lambda_a) \otimes V(\lambda_a)$. The vertex labelled by the pair (c,k) corresponds to the irreducible $U_q[osp(n|n)]$ module $V(k, 2a-2c)$ except for the vertex corresponding to $c = a$, $k = 1$, which has been circled to indicate that it is an indecomposable $U_q[osp(n|n)]$-module.

It can be shown that the extended twisted TPG is consistent, i.e. that the recursion relations (8) give the same result independent of the path along which one recurs. To prove this it suffices to show for each closed loop of four vertices in the graph, that the difference in Casimir eigenvalues for $osp(n|n)$ along one edge equals the difference along the opposite edge.

Let $P_V \equiv P_V^{\lambda_a \lambda_a}$ be the projection operator from $V(\lambda_a) \otimes V(\lambda_a)$ onto V and $P_\nu \equiv P_\nu^{\lambda_a \lambda_a}$ the projector onto $V(\nu)$. Then the R-matrix $\check{R}(z) \equiv \check{R}^{\lambda_a,\lambda_a}(z)$ from the extended twisted TPG can be expanded in terms of the operators N, P_V and P_ν:

$$\check{R}(z) = \rho_N(z)N + \rho_V(z)P_V + \sum_\nu \rho_\nu(z)P_\nu. \tag{24}$$

The coefficients $\rho_\nu(z)$ can be obtained recursively from the extended twisted TPG. However, the coefficients $\rho_N(z)$ and $\rho_V(z)$ cannot be read off from the extended twisted TPG since the corresponding vertex refers to an indecomposable module. Rather they are determined by the approach[16] to $U_q[gl(2|2)^{(2)}]$. The result is[9]

$$\check{R}(z) = \rho_N(z)N + \rho_V(z)P_V + \sum_{c=0}^{a\ '}\sum_{k=0}^{c\ '}\prod_{j=1}^{c-k}\langle 2j-2a\rangle_+ \prod_{i=1}^{c}\langle i-2a-1\rangle_- P_{(2a-2c+k)\delta_1+k\delta_2}, \tag{25}$$

where the primes in the sums signify that terms corresponding to $c=a$ with $k=0,1$ are omitted from the sums, and $\rho_V(z)$, $\rho_N(z)$ are given by

$$\rho_V(z) = \frac{z-q^2}{1-zq^2}\prod_{j=1}^{a-1}\langle 2j-2a\rangle_+ \prod_{i=1}^{a-1}\langle i-2a-1\rangle_-,$$

$$\rho_N(z) = (-1)^a q^{-a^2}\,\frac{1-z}{1+z}\,\rho_V(z). \tag{26}$$

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FREE FIELD AND PARAFERMIONIC REALIZATIONS OF TWISTED $su(3)_k^{(2)}$ CURRENT ALGEBRA

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Free field and twisted parafermionic representations of twisted $su(3)_k^{(2)}$ current algebra are obtained. The corresponding twisted Sugawara energy-momentum tensor is given in terms of three (β, γ) pairs and two scalar fields and also in terms of twisted parafermionic currents and one scalar field. Two screening currents of the first kind are presented in terms of the free fields.

1. Introduction

Infinite dimensional algebras, such as Virasoro algebra and affine algebras are algebraic structures in conformal field theories (CFT) in two dimensional spacetime.[1–3] They also play a central role in the study of string theory.[4]

It is well-known the untwisted current algebra can be realized by at least two different ways: one is the free field representation,[5] and the other is the parafermion representation.[6] The free field realization is a common approach used in conformal field theories.[5] The free field representations for untwisted affine algebra have been extensively studied. The simplest untwisted case $su(2)^{(1)}$ was first treated in Ref. 5, and the generalization to $su(n)^{(1)}$ was given in Refs. 7–17. The Z_k parafermions are generalizations of the Majorana fermions, and the Z_k parafermion models are extensions of the Ising model, which corresponds to the case $k = 2$.[19,20] Parafermions are related to the exclusion statistics introduced by Haldane.[21]

The recently study shows that twisted affine algebras are useful in the description of the entropy of Ads_3 black hole.[18] However, little is known for free field and parafermionic representations of twisted affine algebras. So it is an interesting problem to investigate such realizations.

In this contribution we consider the simplest twisted affine algebra $su(3)_k^{(2)}$. We construct two different realizations for this algebra: one is free field representation,

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which is another version of one given in Ref. 22, and another is twisted parafermionic representation by using the twisted parafermionic currents proposed in Ref. 23. Moreover we give the screening currents in terms of the free fields.

2. Twisted current $su(3)_k^{(2)}$ algebra

We consider the simplest twisted affine Lie algebra $su(3)_k^{(2)}$. We decompose $su(3)$ as

$$su(3) = g_0 \oplus g_1 \tag{1}$$

where $g_0 = su(2)$ is the fixed point subalgebra under the automorphism and g_1 is the five dimensional representation of g_0; g_0 and g_1 satisfy $[g_i, g_j] \subset g_{(i+j) \ mod \ 2}$. Using the notation in Ref. 22, we chose e, f and h to be the bases in g_0 and $\tilde{e}$, $\tilde{f}$, $\tilde{E}$, $\tilde{F}$ and $\tilde{h}$ the bases of g_1. Then the commutators of $su(3)_k^{(2)}$ can be expressed as

$$[z^m \otimes X, z^n \otimes Y] = z^{m+n} \otimes [X, Y] + 2km\delta_{m+n,0}\frac{(X|Y)}{2}. \tag{2}$$

Where $m \in Z$ if $X \in g_0$, and $m \in Z + \frac{1}{2}$ if $X \in g_1$.

Denote the currents corresponding to e, h, f by $j^+(z)$, $j^0(z)$, $j^-(z)$, and to $\tilde{e}$, $\tilde{h}$, $\tilde{f}$, respectively. Then (2) can be written in terms of the following OPE's:

$$j^+(z)j^-(w) = \frac{4k}{(z-w)^2} + \frac{1}{(z-w)}j^0(w) + \ldots,$$

$$J^+(z)J^-(w) = \frac{4k}{(z-w)^2} + \frac{1}{(z-w)}j^0(w) + \ldots,$$

$$J^{++}(z)J^{--}(w) = \frac{4k}{(z-w)^2} + \frac{2}{(z-w)}j^0(w) + \ldots,$$

$$j^0(z)j^{\pm}(w) = \frac{\pm 2}{(z-w)}j^{\pm}(w) + \ldots, \quad j^0(z)j^0(w) = \frac{8k}{(z-w)^2} + \ldots,$$

$$J^0(z)J^{\pm}(w) = \frac{\pm 6}{(z-w)}j^{\pm}(w) + \ldots, \quad J^0(z)j^{\pm}(w) = \frac{\pm 6}{(z-w)}J^{\pm}(w) + \ldots,$$

$$J^+(z)J^{--}(w) = \frac{2}{(z-w)}j^-(w) + \ldots, \quad J^-(z)J^{++}(w) = \frac{-2}{(z-w)}j^+(w) + \ldots,$$

$$j^+(z)J^+(w) = \frac{2}{(z-w)}J^{++}(w) + \ldots, \quad j^-(z)J^-(w) = \frac{-2}{(z-w)}J^{--}(w) + \ldots,$$

$$j^+(z)J^-(w) = \frac{1}{(z-w)}J^0(w) + \ldots, \quad J^+(z)j^-(w) = \frac{1}{(z-w)}J^0(w) + \ldots,$$

$$j^+(z)J^{--}(w) = \frac{-2}{(z-w)}J^-(w) + \ldots, \quad j^-(z)J^{++}(w) = \frac{2}{(z-w)}J^+(w) + \ldots,$$

$$j^0(z)J^{\pm}(w) = \frac{\pm 2}{(z-w)}J^{\pm}(w) + \ldots, \quad j^0(z)J^{\pm\pm}(w) = \frac{\pm 4}{(z-w)}J^{\pm\pm}(w) + \ldots,$$

$$J^0(z)J^0(w) = \frac{24k}{(z-w)^2} + \ldots.$$

All other OPE's contain trivial regular terms only. Here and throughout "..." stands for regular terms.

3. Wakimoto free field realization of the twisted affine currents

To obtain a free field realization of the twisted $su(3)_k^{(2)}$ currents, we follow the procedure adopted in Ref. 22 but begin with a different Fock space. The Fock space is constructed by the repeated actions of f, $\tilde{f}$, $\tilde{F}$ on the highest weight state v_Λ with highest weight Λ. The highest weight states are determined by

$$ev_\Lambda = \tilde{e}v_\Lambda = \tilde{E}v_\Lambda = 0, \quad hv_\Lambda = (\Lambda, \alpha_1)v_\Lambda, \quad \tilde{h}v_\Lambda = (\Lambda, \alpha_2)v_\Lambda \tag{3}$$

where α_1 and α_2 are roots associated with h and $\tilde{h}$.

Set $|n, m, l> = \tilde{f}^n \tilde{F}^m f^l v_\Lambda$. This choice is different from the one used in Ref. 22. As we shall see, this choice gives another free field realization of the twisted current algebra. Introduce three $\beta\gamma$ pairs and two scalar fields ϕ_a, $a = 1, 2$. $(\beta_i;\ \gamma_i)$ pairs have conformal dimension $(1;\ 0)$.

$$\begin{aligned} \beta_i(z)\gamma_j(w) = -\gamma_j(z)\beta_i(w) = -\frac{\delta_{ij}}{z-w}, \qquad i,j = 0,1,2 \\ \phi_a(z)\phi_b(w) = -2\delta_{a,b}\ln(z-w), \qquad a,b = 0,1 \end{aligned} \tag{4}$$

Introduce the notation $\vec{e}_1 = \frac{1}{2}(1,1)$; $\vec{e}_2 = \frac{\sqrt{3}}{2}(1,-1)$; and $\vec{\Phi} = (\phi_0, \phi_1)$. Then we have

$$\vec{e}_1 \cdot \vec{\Phi}(z)\vec{e}_1 \cdot \vec{\Phi}(w) = -\ln(z-w); \qquad \vec{e}_2 \cdot \vec{\Phi}(z)\vec{e}_2 \cdot \vec{\Phi}(w) = -3\ln(z-w);$$
$$\vec{e}_1 \cdot \vec{\Phi}(z)\vec{e}_2 \cdot \vec{\Phi}(w) = 0.$$

We find the Wakimoto free field realization of $su(3)_k^{(2)}$ in terms of the eight free fields:

$$\begin{aligned}
j^+(z) =\ & \beta_0(z) + 2\beta_2(z)\gamma_1(z), \\
j^0(z) =\ & 2\beta_0(z)\gamma_0(z) + 2\beta_1(z)\gamma_1(z) + 4\beta_2(z)\gamma_2(z) + \frac{1}{\alpha_+}(\vec{e}_1 \cdot i\partial\vec{\Phi}(z)), \\
j^-(z) =\ & -\beta_0(z)\left(\gamma_0^2(z) + 3\gamma_1^2(z)\right) - 2\beta_1(z)\left(3\gamma_0(z)\gamma_1(z) - \gamma_2(z)\right) \\
& -2\beta_2(z)\left(3\gamma_0^2(z)\gamma_1(z) + \gamma_1^3(z)\right) - 4(k+1)\partial\gamma_0(z) \\
& -\frac{1}{\alpha_+}(\vec{e}_1 \cdot i\partial\vec{\Phi}(z))\gamma_0(z) - \frac{1}{\alpha_+}(\vec{e}_2 \cdot i\partial\vec{\Phi}(z))\gamma_1(z), \\
J^0(z) =\ & 6\beta_0(z)\gamma_1(z) + 6\beta_1(z)\gamma_0(z) + 6\beta_2(z)\left(\gamma_0^2(z) + \gamma_1^2(z)\right) \\
& + \frac{1}{\alpha_+}(\vec{e}_2 \cdot i\partial\vec{\Phi}(z)); \\
J^+(z) =\ & \beta_1(z), \\
J^-(z) =\ & -2\beta_0(z)\left(\gamma_0(z)\gamma_1(z) + \gamma_2(z)\right) - \beta_1(z)\left(3\gamma_0^2(z) + \gamma_1^2(z)\right) \\
& -4\beta_2(z)\left(\gamma_0^3(z) + \gamma_1(z)\gamma_2(z)\right) - 4k\partial\gamma_1(z), \\
& -\frac{1}{\alpha_+}(\vec{e}_1 \cdot i\partial\vec{\Phi}(z))\gamma_1(z) - \frac{1}{\alpha_+}(\vec{e}_2 \cdot i\partial\vec{\Phi}(z))\gamma_0(z);
\end{aligned} \tag{5}$$

$$
\begin{aligned}
J^{++}(z) = &\ \beta_2(z), \\
J^{--}(z) = &\ 2\beta_0(z)\left(\gamma_0^2(z)\gamma_1(z) + \gamma_1^3(z) - 2\gamma_0(z)\gamma_2(z)\right) \\
&-2\beta_1(z)\left(\gamma_0^3(z) - 3\gamma_0(z)\gamma_1^2(z) + 2\gamma_1(z)\gamma_2(z)\right) \\
&-\beta_2(z)\left(3\gamma_0^4(z) - \gamma_1^4(z) - 6\gamma_0^2(z)\gamma_1^2(z) + 4\gamma_2^2(z)\right) \\
&+\frac{2}{\alpha_+}(\vec{e}_1 \cdot i\partial\vec{\Phi}(z))\left(\gamma_0(z)\gamma_1(z) - \gamma_2(z)\right) - 4k\partial\gamma_2(z) \\
&-\frac{1}{\alpha_+}(\vec{e}_2 \cdot i\partial\vec{\Phi}(z))\left(\gamma_0^2(z) - \gamma_1^2(z)\right) + 8(k+1)\gamma_1(z)\partial\gamma_0(z).
\end{aligned}
$$

Here $\alpha_+ = 1/\sqrt{8k+24}$, and normal ordering is implied in the expressions. It is straightforward to check that the above currents satisfy the OPE given in last section. We remark that the twisted currents have the following mode expansions:

$$
j^a(z) = \Sigma_{n\in Z} j_n^a z^{-n-1}; \qquad J^a(z) = \Sigma_{n\in Z+1/2} J_n^a z^{-n-1}. \tag{6}
$$

.

4. Twisted parafermions and parafermionic realization

In this section we use the twisted parafermionic currents proposed in Ref. 23 to give another realization of the twisted $su(3)_k^{(2)}$ currents. First, we recall the twisted parafermion current algebra given in Ref. 23 reads,

$$
\begin{aligned}
\psi_l(z)\psi_{l'}(w)(z-w)^{ll'/2k} &= \frac{\delta_{l+l',0}}{(z-w)^2} + \frac{\varepsilon_{l,l'}}{z-w}\psi_{l+l'}(w) + \cdots, \\
\psi_{\tilde{l}}(z)\psi_{\tilde{l}'}(w)(z-w)^{ll'/2k} &= \frac{\delta_{\tilde{l}+\tilde{l}',0}}{(z-w)^2} + \frac{\varepsilon_{\tilde{l},\tilde{l}'}}{z-w}\psi_{l+l'}(w) + \cdots, \\
\psi_l(z)\psi_{\tilde{l}'}(w)(z-w)^{ll'/2k} &= \frac{\varepsilon_{l,l'}}{z-w}\psi_{\tilde{l}+\tilde{l}'}(w) + \cdots,
\end{aligned} \tag{7}
$$

where $l,\ l' = \pm 1$ and $\tilde{l},\ \tilde{l}' = \tilde{0}, \pm\tilde{1}, \pm\tilde{2}$; $\varepsilon_{l,l'}$, $\varepsilon_{\tilde{l},\tilde{l}'}$ and $\varepsilon_{l,\tilde{l}'}$ are structure constants. If we denote ψ_l or $\psi_{\tilde{l}}$ by Ψ_a, then we can rewrite the above relations as:

$$
\Psi_a(z)\Psi_b(w)(z-w)^{ab/2k} \equiv \sum_{n=-2}^{\infty}(z-w)^n[\Psi_a\Psi_b]_{-n}, \tag{8}
$$

For consistency, $\varepsilon_{a,b}$ must have the properties: $\varepsilon_{a,b} = -\varepsilon_{b,a} = -\varepsilon_{-a,-b} = \varepsilon_{-a,a+b}$ and $\varepsilon_{a,-a} = 0$. As a CFT, we can set the structure constants as

$$
\begin{aligned}
&\varepsilon_{\tilde{1},-\tilde{2}} = \varepsilon_{1,\tilde{1}} = \varepsilon_{-1,\tilde{2}} = \frac{1}{\sqrt{k}}, \\
&\varepsilon_{\tilde{1},-\tilde{1}} = \varepsilon_{\tilde{0},\tilde{1}} = \sqrt{\frac{3}{2k}}.
\end{aligned} \tag{9}
$$

Then we may obtain a representation of the twisted $su(3)_k^{(2)}$ current algebra with the help of the twisted parafermionic currents. The result is

$$
j^+(z) = 2\sqrt{k}\psi_1(z)e^{\frac{i}{\sqrt{2k}}\phi_0(z)}, \qquad j^-(z) = 2\sqrt{k}\psi_{-1}(z)e^{-\frac{i}{\sqrt{2k}}\phi_0(z)},
$$

$$j^0(z) = 2\sqrt{2k}i\partial\phi_0(z), \qquad J^+(z) = 2\sqrt{k}\psi_{\bar{1}}(z)e^{\frac{i}{\sqrt{2k}}\phi_0(z)},$$
$$J^-(z) = 2\sqrt{k}\psi_{-\bar{1}}(z)e^{-\frac{i}{\sqrt{2k}}\phi_0(z)}, \qquad J^{++}(z) = 2\sqrt{k}\psi_{\bar{2}}(z)e^{i\sqrt{\frac{2}{k}}\phi_0(z)},$$
$$J^{--}(z) = 2\sqrt{k}\psi_{-\bar{2}}(z)e^{-i\sqrt{\frac{2}{k}}\phi_0(z)}, \qquad J^0(z) = 2\sqrt{6k}\psi_{\bar{0}}(z).$$

where ϕ_0 is an $U(1)$ current obeying $\phi_0(z)\phi_0(w) = -\ln(z-w)$. It can be checked that the above currents satisfy the OPEs of the twisted $su(3)_k^{(2)}$ currents algebra.

5. Twisted stress energy tensor

It is well known that Virasoro algebras are related to currents algebras via the so called Sugawara construction. In the present case, the twisted Sugawara construction of the energy-momentum tensor is given by

$$T(z) = \frac{1}{8(k+3)} : \left[\frac{1}{2}j^0(z)j^0(z) + \frac{1}{6}J^0(z)J^0(z) + 2j^-(z)j^+(z) + 2J^-(z)J^+(z) + 2J^{--}(z)J^{++}(z)\right] :, \tag{10}$$

where : : implies the normal ordering. The above expression can be rephrased through the $\beta\gamma$ pairs and the scalar field $\vec{\Phi}$. We obtain

$$T(z) = -:[\beta_0(z)\partial\gamma_0(z) + \beta_1(z)\partial\gamma_1(z) + \beta_2(z)\partial\gamma_2(z)]: + \frac{1}{2}:\left(\vec{e}_1\cdot i\partial\vec{\Phi}(z)\right)^2: + \frac{1}{6}:\left(\vec{e}_2\cdot i\partial\vec{\Phi}(z)\right)^2: - 4\alpha_+\left(\vec{e}_1\cdot i\partial^2\vec{\Phi}(z)\right). \tag{11}$$

On the other hand, the energy-momentum tensor in the twisted parafermionic realization is given by

$$T(z) = T_\psi - :\partial\phi_0(z)\partial\phi_0(z):. \tag{12}$$

where

$$T_\psi = \frac{k}{2k+6}\sum_a [\Psi_a\Psi_{-a}]_0 \tag{13}$$

is the energy-momentum tensor of the twisted parafermion currents obtained in Ref. 23. Following the standard practice, we get the OPE of the energy-momentum tensor,

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots, \tag{14}$$

where $c = 8k/(k+3)$ is the central charge for the Virasoro algebra.

6. Twisted screening currents

An important object in the free field approach is the screening current. Screening currents are a primary fields with conformal dimension 1, and their integration give the screening charges. They commute with the affine currents up to total derivatives. These properties ensure that screening charges may be inserted into correlators while the conformal or affine Ward identities remain intact. For the present case, we find the following screening currents

$$S_{\pm}(z) =: [2\beta_2(z)\gamma_0(z) + \beta_1(z) \pm \beta_0(z)]\, \tilde{S}_{\pm}(z) :, \tag{15}$$

where

$$\tilde{S}_{\pm}(z) = e^{-2\alpha_+\left(\vec{e}_1\cdot i\vec{\Phi}(z) \pm \vec{e}_2\cdot i\vec{\Phi}(z)\right)}. \tag{16}$$

The OPE of the twisted screening currents with the twisted affine currents are

$$\begin{aligned}
T(z)S_{\pm}(w) &= \partial_w\left(\frac{1}{z-w}S_{\pm}(w)\right) + \ldots,\\
j^+(z)S_{\pm}(w) &= \ldots,\\
j^0(z)S_{\pm}(w) &= \ldots,\\
j^-(z)S_{\pm}(w) &= \partial_w\left(\pm\frac{1}{2\alpha_+^2}\frac{1}{z-w}\tilde{S}_{\pm}(w)\right) + \ldots,\\
J^{++}(z)S_{\pm}(w) &= \ldots,\\
J^+(z)S_{\pm}(w) &= \ldots,\\
J^0(z)S_{\pm}(w) &= \ldots,\\
J^-(z)S_{\pm}(w) &= \partial_w\left(\frac{1}{2\alpha_+^2}\frac{1}{z-w}\tilde{S}_{\pm}(w)\right) + \ldots,\\
J^{--}(z)S_{\pm}(w) &= \partial_w\left(\frac{1}{\alpha_+^2}\frac{1}{z-w}\left[\gamma_0(w) \mp \gamma_1(w)\right]\tilde{S}_{\pm}(w)\right) + \ldots;
\end{aligned} \tag{17}$$

The screening currents obtained here are the twisted versions of the first kind screening currents.[7]

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